The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2016-2017**

Operations Research **2 Hours**

Attempt all FOUR questions.

1 (i) You are given the following linear programming problem:

$$\max \quad z = 2x_1 - x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 6, \qquad 2x_1 - 2x_2 \leq 1.$$

(a) Let x_3 and x_4 be the slack variables, write down the canonical form of the problem. ***(2 marks)***

(b) Construct the initial tableau, and identify the basic and nonbasic variables. ***(2 marks)***

(c) After a few steps of simplex iterations, the following tableau is found:

Basis	x_1	x_2	x_3	x_4	Solution
z	0	-1	0	1	1
x_3	0	2	1	$-1/2$	$11/2$
x_1	1	-1	0	$1/2$	$1/2$

Is the tableau already optimal? Give your reasons. If it is not, use the simplex method to continue the calculation and find the optimal solution. ***(6 marks)***

(ii) We use the two-phase method to solve the following linear programming problem:

$$\max \quad z = 3x_1 - 5x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + 2x_2 \leq 6, \quad 3x_1 + x_2 \geq 2, \quad x_1 - x_2 \leq 1.$$

Construct the initial tableau in phase 1 and preprocess it so that it is suitable to be solved using the simplex method. **Do NOT proceed further.** ***(7 marks)***

(iii) A company manufactures three products. The daily labour and raw material requirements for the products and their availability are given in the following table.

Product	Labour (hr/unit)	Cost of raw material £/unit
1	3	4
2	4	3
3	5	6
Availability	140	100

The profit per unit is £27, £30, and £35, respectively. If product 3 is to be manufactured, then its production level must be at least 10 units. Formulate the mixed integer-linear programming problem you may solve to find the maximum total profit. **Find the formulation only; do NOT attempt to solve it**. ***(8 marks)***

2 (i) Chinaco makes three types of tableware: plates, cups, and bowls. The table below provides the pertinent data regarding the use of raw material and labour time together with profit estimates.

	Plates	Cups	Bowls
Cost of raw material per unit (£)	5	3	8
Labour time per unit (hrs)	4	3	5
Profit per unit (£)	3	2	2

Besides, it is known that:

- The available capital for raw material is estimated at £3000 and available labour time is limited to 2500 hours.
- A one time setup cost is required if a product is to be made. The setup cost for making the plates is £100. Cups and bowls are made by another machine that costs £80 to set up. Since cups and bowls are made by the same machine, the machine needs to be set up only once if both are to be produced.

Formulate the mixed integer linear programming model you may use to find the optimum number of units that Chinaco should manufacture of each product. **Find the formulation only; do NOT attempt to solve it**.

(14 marks)

(ii) As the owner of a small shop, you are making plans to purchase a product from a supplier, which you will sell to your customers.

- Based on market projections, it is assumed that you will sell 300, 320, 450, and 250 units of the product in the next four quarters, respectively.
- The price per unit to **purchase from the supplier** starts at £20 in the first quarter and increases by £2 each quarter thereafter.
- The supplier can provide no more than 400 units in any one quarter.
- Although we can take advantage of lower prices in early quarters, a storage cost of £7 is incurred per unit per quarter if the product needs to be held over from one quarter to the next. No storage cost is incurred if the product is purchased and sold in the same quarter.

Let x_1, x_2, x_3, and x_4 be the numbers of the product to be purchased in the quarters 1, 2, 3, and 4, and s_1, s_2 and s_3 be the numbers remained in storage after quarter 1, 2, 3, respectively. Develop a linear programming model to minimize the total expense on purchase and storage. **Find the formulation only; do NOT attempt to solve it**. ***(11 marks)***

3 We consider a primal linear programming problem

$$\max\ z = c^T x, \quad \text{subject to} \quad Ax \geq b \quad \text{and} \quad x \geq 0. \tag{1}$$

The dual prolem is given as follows:

$$\min\ v = -y^T b, \quad \text{subject to} \quad -A^T y \geq c \quad \text{and} \quad y \geq 0. \tag{2}$$

(i) Write down the Lagrangian functions for the primal and the dual problems, respectively. ***(4 marks)***

(ii) Show that the dual problem of the problem defined by Equation (2) is the same as the problem defined by Equation (1). ***(11 marks)***

(iii) We introduce suitable slack variables for the primal problem defined by Equation (1). Thus we can re-write the constraint as $\overline{A}x = b$, where $\overline{A} = [A, -I]$, and x denotes the vector of both the decision and the slack variables. Let $y^+ = -B^{-T}c_B$, where B is the optimal basic matrix for the primal problem, and c_B is the corresponding cost coefficient vector. Show that y^+ is a **feasible** solution for the dual problem defined by Equation (2). ***(10 marks)***

4 A company produces four types of trolleys: T1, T2, T3, and T4. To make one trolley, certain amounts of machine time and worker-days are needed. Relevant information is given in the following table:

	T1	T2	T3	T4	Availability
Machine time (hours/unit)	1	3	8	4	90
Worker-days (per unit)	1	1	1	3	80
Profit (£/unit)	1	2	4	3	

To find the optimal production schedule, we let x_1, x_2, x_3 and x_4 be the numbers of trolleys T1, T2, T3 and T4 to be manufactured, and x_5 and x_6 be the slack variables corresponding to the machine time and worker-day constraints, respectively. The linear programming model is formulated as follows:

$$\begin{aligned} \text{max} \quad & z = x_1 + 2x_2 + 4x_3 + 3x_4\,, \\ \text{subject to} \quad & x_1 + 3x_2 + 8x_3 + 4x_4 \leq 90\,, \\ & x_1 + x_2 + x_3 + 3x_4 \leq 80\,. \end{aligned}$$

Using the simplex method, the optimal tableau has been determined to be:

	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	1/2	1/2	1/2	1/2	85
x_2	0	1	7/2	1/2	1/2	−1/2	5
x_1	1	0	−5/2	5/2	−1/2	3/2	75

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables, the cost function, and the optimal solutions for the dual variables. ***(3 marks)***

(ii) Using the information given by the above tableau, verify that the complementary slackness conditions are satisfied for both the above problem and its dual problem. ***(4 marks)***

(iii) Determine the range of the unit profit for trolley T1 which leaves the current optimal basis unchanged. ***(8 marks)***

(iv) The company is considering manufacturing one more type of trolley. To make one unit of this type of trolley requires 1 hour of machine time and 2 worker-days. The profit for this unit is £3. Determine whether the company should indeed manufacture this type of trolley. ***(4 marks)***

(v) Find the range of values for available worker-days for which the optimal basis remains the same. ***(3 marks)***

(vi) Thanks to technology innovation, it is now possible to make trolley T3 with only 4 hours of machine time and 2 worker-days. If this innovation is adopted, does the solution remain optimal? Give your reasons. ***(3 marks)***

End of Question Paper