The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016-2017**

Milestones in Applied Mathematics II **2 Hours**

Answer all four questions.

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1 An operator A is defined by

$$A = \sqrt{\frac{m\omega}{2\hbar}}\, x + i\frac{p}{\sqrt{2m\hbar\omega}}.$$

where p and x are the momentum and position operator respectively.

(i) Calculate the adjoint operator A^* of A.

(2 marks)

(ii) For the energy operator H defined by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

show that

$$H = \hbar\omega\left(A^*A + \frac{1}{2}\right).$$

(6 marks)

(iii) Hence, show that

$$[H, A^*] = \hbar\omega A^*.$$

(7 marks)

(iv) You are given that u_n is the energy eigenfunction of H with the energy eigenvalue E_n where $n = 0, 1, 2.....$ Show that A^*u_n is an energy eigenfunction. Calculate the corresponding energy eigenvalue.

(5 marks)

(v) One unnormalised energy eigenfunction is

$$\psi_n = (2x^3 - 3x)\exp\left(-x^2/2\right).$$

Let $m = \omega = \hbar = 1$ for this part and find the (unnormalised) eigenfunction which has larger energy and is closest in energy to ψ_n.

(5 marks)

2 A free particle of mass m is confined to the one-dimensional region $0 \le x \le a$. At $t = 0$, its normalised wave function is

$$\psi(x, t=0) = \sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right)\left[1 + 2\cos\left(\frac{\pi x}{a}\right)\right].$$

(i) Calculate normalised stationary state wave functions.

(9 marks)

(ii) Show that the wave function at a later time $t = t_0$ is given by

$$\psi(x, t_0) = \sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right)\left[\exp\left(-\frac{i\pi^2\hbar t_0}{2ma^2}\right) + \exp\left(-\frac{2i\pi^2\hbar t_0}{ma^2}\right)\cos\left(\frac{\pi x}{a}\right)\right].$$

(6 marks)

(iii) What is the average energy of the system at $t = 0$ and $t = t_0$? ***(5 marks)***

(iv) Find the probability that at time t the particle lies within the interval $[0, \frac{a}{2}]$ at $t = t_0$.

(5 marks)

3 (i) A free particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is in an energy eigenstate

$$\psi(x) = (\gamma^2/\pi)^{1/4} \exp\left(-\gamma^2 x^2/2\right),$$

with energy $E = \hbar^2\gamma^2/2m$. Find the mean position of the particle, the mean momentum of the particle, and $V(x)$.

(15 marks)

(ii) A free electron with mass is 9.1×10^{-31}Kg is moving with the speed 3×10^5 m/s. Calculate the de Broglie wavelength of this electron. You are given that the Planck constant is $h = 6.626 \times 10^{-34}$ J s.

(5 marks)

(iii) The threshold frequency of light required to eject electrons from the surface of certain metal is 4×10^{14} Hz. What is the work function of the metal?

(3 marks)

(iv) For operators A, B and C, show $[A + B, C] = [A, C] + [B, C]$.

(2 marks)

4 (i) Let A and B be self-adjoint operators. Show that AB is not self-adjoint in general, but that $AB + BA$ is self-adjoint.

(5 marks)

(ii) Consider a one-dimensional system with potential energy on the potential step

$$V(x) = \begin{cases} V_0 & x > 0; \\ 0 & x < 0, \end{cases}$$

where V_0 is a positive constant. A beam of particles with energy E is incident from the left (i.e. from $x = -\infty$).

(a) Write down the time-independent Schrödinger equation for $x < 0$ and $x > 0$. Hence, solve the Schrödinger equations for $E < V_0$.

(12 marks)

(b) Hence, for $E < V_0$, calculate the fraction of the beam that is transmitted and the fraction of the beam that is reflected.

(8 marks)

End of Question Paper