Mathematical Methods

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

1. The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \mathcal{F} \{ f(x) \} = \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx.$$ 

(a) Using the above definition, derive the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) \, dk. \quad (8 \text{ marks})$$

You may assume that $\int_{-\infty}^{\infty} e^{ikx} \, dk = 2\pi \delta(x)$.

(b) Show that the Fourier transform of

$$f(x) = x \, e^{-|k|}$$

is given by

$$\hat{f}(k) = \frac{4ik}{(1 + k^2)^2}. \quad (10 \text{ marks})$$

By applying the inverse Fourier transform to $\hat{f}(k)$, deduce that for real $x$

$$\int_{0}^{\infty} \frac{k \sin kx}{(1 + k^2)^2} \, dk = \frac{\pi}{4} x \, e^{-|k|}. \quad (7 \text{ marks})$$
The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} + 2\dot{x} + 2x = f(t)$$

for $t \geq 0$, for some function $f(t)$, with $x(0) = -1$ and $\dot{x}(0) = 3$.

(a) Taking the Laplace transform of the equation, find $\tilde{x}(s)$ in terms of $\tilde{f}(s)$, where the Laplace transform $\tilde{x}(s)$ is defined by

$$\tilde{x}(s) = \int_0^\infty e^{-st} x(t) \, dt,$$

and $\tilde{f}(s)$ is defined similarly. (5 marks)

Hence derive the solution

$$x(t) = e^{-t}(2\sin t - \cos t) + \int_0^t f(u) e^{-(t-u)} \sin(t-u) \, du. \quad (5 \text{ marks})$$

You may assume that the following hold:

$$\begin{align*}
\mathcal{L} \{x^{(n)}(t)\} &= s^n \tilde{x}(s) - s^{n-1} x(0) - s^{n-2} \dot{x}(0) - \cdots - x^{(n-1)}(0) \\
\mathcal{L} \{e^{at} g(t)\} &= \tilde{g}(s-a) \\
\mathcal{L} \{\sin \omega t\} &= \frac{\omega}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L} \{\cos \omega t\} = \frac{s}{s^2 + \omega^2} \\
\mathcal{L} \left\{ \int_0^t f(u) g(t-u) \, du \right\} &= \tilde{f}(s)\tilde{g}(s),
\end{align*}$$

where $\mathcal{L} \{\cdot\}$ denotes the Laplace transform.

(b) Use the result of part (a) to find the solution $x(t)$ when $f(t) = e^{2t}$. (8 marks)

Verify that this solution does satisfy the differential equation and the initial conditions. (7 marks)
3. The function \( y(x) \) satisfies the ordinary differential equation

\[
y'' - y = \ln(1 + x) \quad 0 \leq x \leq 1, \tag{1}
\]

with the boundary conditions

\[
y = 0 \quad \text{at } x = 0 \quad \text{and at } x = 1.
\]

(a) Find the independent solutions of

\[
y'' - y = 0. \tag{3 \text{ marks}}
\]

(b) Given that Green’s function \( G(x; \xi) \) for the boundary-value problem given at the beginning of the question is continuous at \( x = \xi \), and that \( \frac{\partial G}{\partial x} \) has a discontinuity of size 1 at \( x = \xi \), show that

\[
G(x; \xi) = \begin{cases} 
\frac{\sinh(\xi - 1) \sinh x}{\sinh 1} & 0 \leq x < \xi, \\
\frac{\sinh \xi \sinh(x - 1)}{\sinh 1} & \xi < x \leq 1.
\end{cases} \tag{14 \text{ marks}}
\]

(c) Use Green’s function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the \( \xi \) integrals). \tag{3 \text{ marks}}

Use this to find \( y'(x) \), and hence to show that

\[
y'(0) = \frac{1}{\sinh 1} \int_0^1 \sinh(\xi - 1) \ln(1+\xi) \, d\xi. \tag{5 \text{ marks}}
\]
Consider the equation

$$\varepsilon x^3 - 2x^2 + 18 = 0,$$  

where $\varepsilon$ is a constant satisfying $0 < \varepsilon \ll 1$.

(a) The solution can be written as

$$x = \frac{1}{\varepsilon} \left( x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\varepsilon \to 0$.

Substitute into equation (2) to derive the solutions for $x$, correct to order $\varepsilon$ as $\varepsilon \to 0$. \hspace{1cm} (19 marks)

(b) Given the rearrangement

$$x = \frac{2}{\varepsilon} \left( 1 - \frac{9}{x^2} \right),$$

of (2), use iteration to find the solution close to $\frac{2}{\varepsilon}$, correct to order $\varepsilon^3$ as $\varepsilon \to 0$. \hspace{1cm} (6 marks)
The exponential integral is defined by

\[ E(x) = \int_1^\infty t^{-1} e^{-xt} \, dt \quad \text{for} \quad x > 0. \]

(a) Show, by changing variables, that

\[ e^x E(x) = \int_0^\infty \frac{e^{-xv}}{1 + v} \, dv. \quad (3 \text{ marks}) \]

Use the sum of a geometric progression to show that

\[ \frac{1}{1 + v} = 1 - v + v^2 - v^3 + \cdots + (-v)^{n-1} + \frac{(-v)^n}{1 + v}. \quad (3 \text{ marks}) \]

By using the above results and considering

\[ I_n(x) = \int_0^\infty v^n e^{-xv} \, dv \quad \text{for} \quad n = 0, 1, 2, \ldots \]

deduce that

\[ e^x E(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^{n-1}(n - 1)!}{x^n} + R_n(x), \]

where

\[ R_n(x) = (-1)^n \int_0^\infty \frac{v^n e^{-xv}}{1 + v} \, dv. \quad (11 \text{ marks}) \]

(b) By considering

\[ \left| \frac{R_n(x)}{(-1)^{n-1}(n-1)!} \right| \]

as \( x \to \infty \), show that \( E(x) \) has the asymptotic series

\[ E(x) \sim e^{-x} \left( \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^n n!}{x^{n+1}} + \cdots \right) \]

as \( x \to \infty \). \quad (8 \text{ marks})