The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2016–2017

Mathematical Methods — **2 hours**

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

1 The Fourier transform, $\hat{f}(k)$, of a function $f(x)$ is defined by

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

(a) Using the above definition, derive the result

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\hat{f}(k)\, dk.$$ ***(8 marks)***

$$\left[\text{You may assume that} \quad \int_{-\infty}^{\infty} e^{ikx}\, dk = 2\pi\delta(x).\right]$$

(b) Show that the Fourier transform of

$$f(x) = x\, e^{-|x|}$$

is given by

$$\hat{f}(k) = \frac{4ik}{(1+k^2)^2}.$$ ***(10 marks)***

By applying the inverse Fourier transform to $\hat{f}(k)$, deduce that for real x

$$\int_0^{\infty} \frac{k\sin kx}{(1+k^2)^2}\, dk = \frac{\pi}{4} x\, e^{-|x|}.$$ ***(7 marks)***

2 The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} + 2\dot{x} + 2x = f(t)$$

for $t \geqslant 0$, for some function $f(t)$, with $x(0) = -1$ and $\dot{x}(0) = 3$.

(a) Taking the Laplace transform of the equation, find $\tilde{x}(s)$ in terms of $\tilde{f}(s)$, where the Laplace transform $\tilde{x}(s)$ is defined by

$$\tilde{x}(s) = \int_0^\infty e^{-st}\, x(t)\, dt,$$

and $\tilde{f}(s)$ is defined similarly. ***(5 marks)***

Hence derive the solution

$$x(t) = e^{-t}(2\sin t - \cos t) + \int_0^t f(u)\, e^{-(t-u)} \sin(t-u)\, du.$$ ***(5 marks)***

You may assume that the following hold:

$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n \tilde{x}(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \cdots - x^{(n-1)}(0)$$

$$\mathcal{L}\left\{e^{at}g(t)\right\} = \tilde{g}(s-a)$$

$$\mathcal{L}\left\{\sin \omega t\right\} = \frac{\omega}{s^2+\omega^2} \quad \text{and} \quad \mathcal{L}\left\{\cos \omega t\right\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = \tilde{f}(s)\tilde{g}(s),$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.

(b) Use the result of part (a) to find the solution $x(t)$ when $f(t) = e^{2t}$. ***(8 marks)***

Verify that this solution does satisfy the differential equation and the initial conditions. ***(7 marks)***

3 The function $y(x)$ satisfies the ordinary differential equation

$$y'' - y = \ln(1+x) \qquad 0 \leqslant x \leqslant 1, \tag{1}$$

with the boundary conditions

$$y = 0 \quad \text{at} \quad x = 0 \quad \text{and at} \quad x = 1.$$

(a) Find the independent solutions of

$$y'' - y = 0.$$ ***(3 marks)***

(b) Given that Green's function $G(x;\xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size 1 at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \dfrac{\sinh(\xi-1)\,\sinh x}{\sinh 1} & 0 \leqslant x < \xi, \\[2ex] \dfrac{\sinh\xi\,\sinh(x-1)}{\sinh 1} & \xi < x \leqslant 1. \end{cases}$$ ***(14 marks)***

(c) Use Green's function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the ξ integrals). ***(3 marks)***

Use this to find $y'(x)$, and hence to show that

$$y'(0) = \frac{1}{\sinh 1}\int_0^1 \sinh(\xi-1)\,\ln(1+\xi)\,d\xi.$$ ***(5 marks)***

4 Consider the equation

$$\epsilon x^3 - 2x^2 + 18 = 0, \tag{2}$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution can be written as

$$x = \frac{1}{\epsilon}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Substitute into equation (2) to derive the solutions for x, correct to order ϵ as $\epsilon \to 0$. ***(19 marks)***

(b) Given the rearrangement

$$x = \frac{2}{\epsilon}\left(1 - \frac{9}{x^2}\right)$$

of (2), use iteration to find the solution close to $\frac{2}{\epsilon}$, correct to order ϵ^3 as $\epsilon \to 0$. ***(6 marks)***

5 The *exponential integral* is defined by

$$E(x) = \int_1^\infty t^{-1}\, e^{-xt}\, dt \qquad \text{for} \qquad x > 0.$$

(a) Show, by changing variables, that

$$e^x E(x) = \int_0^\infty \frac{e^{-xv}}{1+v}\, dv.$$ ***(3 marks)***

Use the sum of a geometric progression to show that

$$\frac{1}{1+v} = 1 - v + v^2 - v^3 + \cdots + (-v)^{n-1} + \frac{(-v)^n}{1+v}.$$ ***(3 marks)***

By using the above results and considering

$$I_n(x) = \int_0^\infty v^n\, e^{-xv}\, dv \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

deduce that

$$e^x E(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + R_n(x),$$

where

$$R_n(x) = (-1)^n \int_0^\infty \frac{v^n\, e^{-xv}}{1+v}\, dv.$$ ***(11 marks)***

(b) By considering

$$\left| \frac{R_n(x)}{\frac{(-1)^{n-1}(n-1)!}{x^n}} \right|$$

as $x \to \infty$, show that $E(x)$ has the asymptotic series

$$E(x) \sim e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots + \frac{(-1)^n\, n!}{x^{n+1}} + \cdots \right)$$

as $x \to \infty$. ***(8 marks)***

End of Question Paper