Data provided: Formulae sheet

The University Of Sheffield.

MAS340

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2016-2017

Mathematics (Computational Methods) — Two hours

Attempt all FOUR questions

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic:

(a) $$6\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 4\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$$ ***(1 mark)***

(b) $$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 6$$ ***(1 mark)***

(c) $$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 3\frac{\partial u}{\partial y}$$ ***(1 mark)***

(ii) Use Taylor series expansions to derive the following approximations,

$$\left(\frac{dy}{dx}\right)_{x=x_0} \approx \frac{y_1 - y_0}{h}, \qquad \left(\frac{d^2y}{dx^2}\right)_{x=x_0} \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2},$$

where $y = y(x)$ is a continuous function and $y_i = y(x_0 + ih)$, $i = 0, \pm 1$. In each case state the order of the error term. ***(6 marks)***

(iii) (a) Derive the standard five-point difference scheme for the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -3(x^2 + y^2)$$

at node points $(x_i, y_j) = (ih, jk)$ of a square grid with mesh-size $h = k = 1/2$. ***(4 marks)***

(b) Given the above differential equation is satisfied on the square $-1 \leq x, y \leq 1$ together with the boundary conditions

$$u(x, y) = 0 \quad \text{on } x = -1 \text{ and } x = 1,$$

$$\frac{\partial u}{\partial y} = u \text{ on } y = -1$$

and

$$\frac{\partial u}{\partial y} = -u \text{ on } y = 1$$

derive the equations for the unknown nodal values. You should draw a diagram with the nodes clearly labelled and make full use of symmetry.
YOU ARE **NOT** REQUIRED TO SOLVE THESE EQUATIONS.
(12 marks)

2 (i) Let $a = x_0 < x_1 < \cdots < x_N = b$, and let $f(x_i) = f_i$ for function f which is continuous on $[a, b]$. Describe the properties of the cubic spline interpolant S to f at the points $x_i,\ i = 0, 1, \cdots, N$. ***(3 marks)***

(ii) Let the cubic interpolant have the form

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$

defined on the interval $[x_i, x_{i+1}]$.
Using the substitution $s''(x_i) = \sigma_i \qquad (0 \leq i \leq n)$, where $'$ denotes differentiation with respect to x, show that

$$c_i = \frac{f_{i+1} - f_i}{h} - \frac{h\sigma_i}{2} - h^2 a_i,$$

where $h = x_{i+1} - x_i$ and f_i is the value of the function at the i-th position. ***(10 marks)***

(iii) Using the conditions derived in (ii), determine the natural cubic spline between the following data points

x	1	2	3	4
f(x)	1	1/2	1/3	1/4

(8 marks)

(iv) Use the cubic spline interpolant to estimate $f(2.5)$ and $f'(2.5)$. ***(4 marks)***

3 Using standard notation, the explicit Dynamical Programming formulation of the Cargo-Loading problem can be expressed as

$$F_j(y_j) = max_{k_j=0,\ldots,K_j} \left\{ v_j k_j + F_{j+1}\left(y_j - w_j k_j\right) \right\},$$

where $K_j \equiv [{}^{y_j}/{}_{w_j}]$, that is, K_j is the integer part of ${}^{y_j}/{}_{w_j}$.
A ship can carry 4 units of weight. Three possible types of object may be transported, each having weight and value as shown in the table.

	Type j	**Weight** w_j	**Value** v_j
Stage 1	1	2	75
Stage 2	2	3	50
Stage 3	3	1	35

Use the Dynamical Programming algorithm to find the maximum possible value of the cargo and determine the optimal solution. ***(25 marks)***

4 (i) Show analytically that the function

$$f(x, y) = 2(x + y)^2 + (x - y)^2 + 3x + 2y$$

has a minimum at the point $(-7/16, -3/16)$. ***(6 marks)***

(ii) Starting at the point $(0, 0)$ perform one iteration of the steepest-descent algorithm to determine an approximation to the minimum point. Do your working to FOUR decimal places. ***(12 marks)***

(iii) Now calculate an approximation to the minimum of $f(x, y)$ using one iteration of Newton's method starting from the point $(0, 0)$. ***(7 marks)***

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$