The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016-2017**

Applicable Analysis **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

You may use the following results when answering questions on this paper.

Table of Laplace Transforms	
Function	*Laplace Transform*
$t^{\alpha}e^{bt}(\alpha > -1)$	$\dfrac{\Gamma(\alpha+1)}{(s-b)^{\alpha+1}}$
$\sin at$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\dfrac{s}{s^2+a^2}$
$f(t)e^{bt}$	$F(s-b)$
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}$
$tf(t)$	$-F'(s)$

1 (i) Define what is meant by the statement that $\int_a^\infty f(x)\,dx$ exists.

(2 marks)

What is meant by the statement that $\int_a^\infty f(x)\,dx$ does not exist. ***(1 mark)***

By using your definition, decide whether the following integral exists

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$ ***(3 marks)***

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form $\int_a^\infty f(x)\,dx$. Your statement should include conditions under which the results are valid. ***(4 marks)***

(iii) Determine whether each of the following integrals converge, giving reasons for your answer:

(a) $\int_0^\infty xe^{-2x}\cos x\,dx,$ ***(4 marks)***

(b) $\int_0^1 \frac{e^{-x}}{x^2\sqrt{x}}\,dx.$ ***(4 marks)***

(iv) Determine whether $\int_0^\infty \frac{\cos x}{\sqrt{x^2+1}}\,dx$ converges or diverges, giving reasons for your answer. ***(7 marks)***

2 (i) State, without proof, the theorem concerning differentiation of an integral of the form $\int_a^\infty f(x,y)\,dx$. Your statement should include conditions under which the result holds. ***(4 marks)***

(a) Let

$$F(y) \quad = \quad \int_0^\infty 4e^{-x^2 y}\,dx \quad (y > 0)\,.$$

Prove that F is differentiable on every interval $[c,d]$ with $0 < c < d$; ***(6 marks)***

(b) Show also that,

$$F'(y) = -\frac{1}{2y}\,F(y) \qquad (*)$$

for $c \le y \le d$. ***(3 marks)***

Deduce that $(*)$ holds for all $y > 0$. ***(1 mark)***

By solving the differential equation $(*)$, find an expression for $F(y)$ in terms of y valid for $y > 0$. You may assume that $\int_0^\infty e^{-t^2}dt \quad = \quad \frac{1}{2}\sqrt{\pi}\,.$ ***(5 marks)***

(ii) Define the Gamma function. ***(2 marks)***

Prove that

$$\int_0^\infty x^2 e^{-3x^2}\,dx \quad = \quad \frac{\sqrt{\pi}}{12\sqrt{3}}\,.$$ ***(4 marks)***

3 (i) Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. *(3 marks)*

Prove that

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \qquad (x>0,\ y>0)$$

and

$$B(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(1+u)^{x+y}} \, du \qquad (x>0,\ y>0).$$

(2 marks)

(ii) Prove each of the following, stating any standard results you need to use:

(a)

$$\int_0^{\pi/2} \frac{\sin^3\theta}{\sqrt{\cos\theta}} \, d\theta = \frac{8}{5}.$$

(b)

$$\int_{-\infty}^{\infty} \frac{e^{3x}}{(e^{4x}+1)^2} \, dx = \frac{\pi\sqrt{2}}{16} .$$

(12 marks)

(iii) Find the area A enclosed by the curve

$$|x^6| + |y^3| = 1.$$

You should simplify as far as possible, expressing your answer in terms of values of the Gamma Function. *(8 marks)*

4 (i) In each of the following cases, find the function continuous on $[0, \infty)$, with the Laplace transform given below:

(a)

$$\frac{6}{s^2 - 9} \quad (s > 3)\,,$$

(b)

$$\frac{2s+1}{s^2 + 2s + 2} \quad (s > -1)\,.$$

(5 marks)

(ii) (c) Suppose the functions f and g are continuous on $[0, \infty)$. Define the convolution $f * g$.

State, without proof, a relation between $L(f * g)$, $L(f)$ and $L(g)$. ***(3 marks)***

(iii) Find the function f continuous on $[0, \infty)$ such that

$$f'(t) + \int_0^t f(u)\,(t-u)\,du \quad = \quad 1 \quad (t \geq 0)\,.$$

and $f(0) \;=\; 4$. ***(17 marks)***

5 (i) (a) Let $f : [0, \infty) \to \mathbb{R}$ be continuous and suppose that the Laplace transform $F = L(f)$ exists on (c, ∞) for some $c \in \mathbb{R}$. State, without proof, the formula giving $L\left(\frac{f(t)}{t}\right)$ in terms of F. Your statement should include sufficient conditions to ensure the validity of the formula. ***(2 marks)***

(b) Prove that the Laplace transform of $\dfrac{\cos t - e^{-t}}{t}$ is given by the formula

$$L\left(\frac{\cos t - e^{-t}}{t}\right) = -\tfrac{1}{2}\ln(1+s^2) + \ln(1+s) \, .$$ ***(7 marks)***

(ii) Verify that, for $s \neq -1$,

$$\frac{2}{(s^2+1)(s+1)^2} = \frac{1}{(s+1)^2} + \frac{1}{(s+1)} - \frac{s}{s^2+1} \, .$$ ***(1 mark)***

Hence find the solution of the differential equation

$$t\,\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = -\,2te^{-t} \quad (t \geq 0)$$

such that $y(0) = 0$. ***(15 marks)***

End of Question Paper