The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.
Strings, pipe cleaners, shoe laces or similar aids for making knots may be used.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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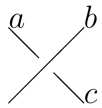
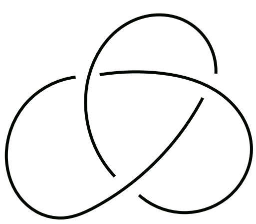
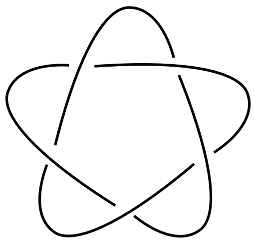
1 (i) Draw the three *Reidemeister moves* and state *Reidemeister's Theorem.* Explain briefly the importance of Reidemeister's Theorem. ***(7 marks)***

(ii) Suppose that $n \geq 2$ is an integer. An n-colouring of a link diagram consists of assigning one of the numbers $0, 1, 2, \ldots, n-1$ to each arc in the diagram so that more than one of the numbers is used and so that at each crossing we have $a + c \equiv 2b \mod n$.



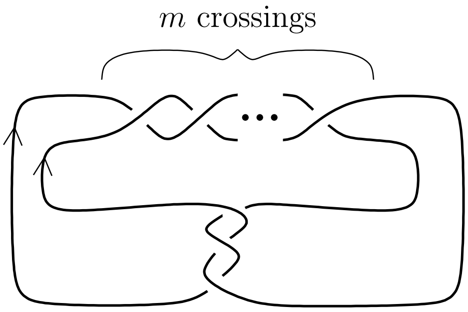
(a) Show that the trefoil diagram T is 3-colourable and that the quinquefoil diagram Q is 5-colourable. ***(4 marks)***

$T =$  $Q =$ 

(b) Show that, when n is fixed, n-colourability is invariant under R-equivalence (i.e. that if two link diagrams are R-equivalent, either both are n-colourable or neither is). ***(10 marks)***

(c) Use the idea of n-colourability to show that the trefoil knot and the quinquefoil knot are both knotted and are *not* equivalent to each other. ***(4 marks)***

2 (i) Let a_m be the illustrated oriented link diagram with $m + 3$ crossings (for $m \geq 0$).

$a_m =$ 

(a) Draw a_0, a_1, a_2 and a_3. Identify the links a_0 and a_1, and hence write down the Jones polynomials $f[a_0]$ and $f[a_1]$. ***(6 marks)***

(b) Show that for $m \geq 2$, the Jones polynomial satisfies the equation

$$f[a_m] = A^2(A^4 - 1)f[a_{m-1}] + A^8 f[a_{m-2}].$$

(5 marks)

(c) Show by induction on m that, for $m \geq 0$,

$$f[a_m] = \frac{A^{2m-12}\left\{A^{4m}(A^{12} - A^8 - 1) + (-1)^m(A^{16} + A^8 + 1)\right\}}{A^4 + 1}.$$

(5 marks)

(d) Find the maximum degree and minimum degree of A in $f[a_m]$ for $m \geq 0$. ***(5 marks)***

(e) State a condition satisfied by the Jones polynomial of an amphicheiral link and hence find the values of m for which a_m is amphicheiral. ***(4 marks)***

3 State whether the following are true or false. *Carefully* justify your answer with a proof or counterexample as necessary. Most of the marks are awarded for the justification.

(i) The Klein bottle K is oriented. ***(5 marks)***

(ii) If L, M and N are compact, connected surfaces, with $M\#L \cong N\#L$, then $M \cong N$. ***(5 marks)***

(iii) There are planes P, Q and R in $\mathbb{R}^3$ such that $P \cup Q$ is a surface whereas $P \cup R$ is not a surface. ***(5 marks)***

(iv) The words $abca^{-1}b^{-1}c^{-1}$ and $abcc^{-1}b^{-1}a^{-1}$ are word equivalent. ***(5 marks)***

(v) Suppose that γ is a path drawn on an oriented surface such that γ returns to its starting point and does not cross itself, then γ separates the surface into two pieces. ***(5 marks)***

4 (i) (a) State the *inclusion/exclusion principle* for the Euler characteristic, and state the Euler characteristic of a *point*, a *line segment* and a *circle*. ***(6 marks)***

(b) Using these together with the homeomorphism invariance of the Euler characteristic, find the Euler characteristic of a *tube* and of a *torus*. ***(8 marks)***

(ii) (a) State the formula for the Euler characteristic of surface in terms of a covering pattern of the surface, explaining any symbols you use. ***(3 marks)***

(b) Suppose that you wish to make a polyhedral convex ball from hexagons and pentagons. Show that you need to use exactly 12 pentagons. What is the minimal number of hexagons needed? Justify your answer. ***(8 marks)***

End of Question Paper