The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Additional Material: *Diagram for Question 2*

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Do not remove it from the hall

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1 (i) Define the set Isom_2 of isometries of $\mathbf{R}^2$ and show that it is closed under composition and taking inverses. ***(7 marks)***

(ii) Define $\psi : \mathrm{Isom}_2 \to O_2$ and show that it is a homomorphism. ***(3 marks)***

(iii) Given points $a, b \in \mathbf{R}^2$ and angles θ and ϕ, show that $R_{a,\theta}R_{b,\phi}R_{a,\theta}^{-1}R_{b,\phi}^{-1} = T_d$ for some vector d. ***(3 marks)***

(You **need not** prove that

$$d = (1 - R_\theta)(1 - R_\phi)(a - b).$$

However, you may use this fact in the next part of this question.)

(iv) Let H be a subgroup of Isom_2, and suppose that $\mathrm{Trans}(H) = \{0\}$ and $|\psi(H)|$ is finite and odd.

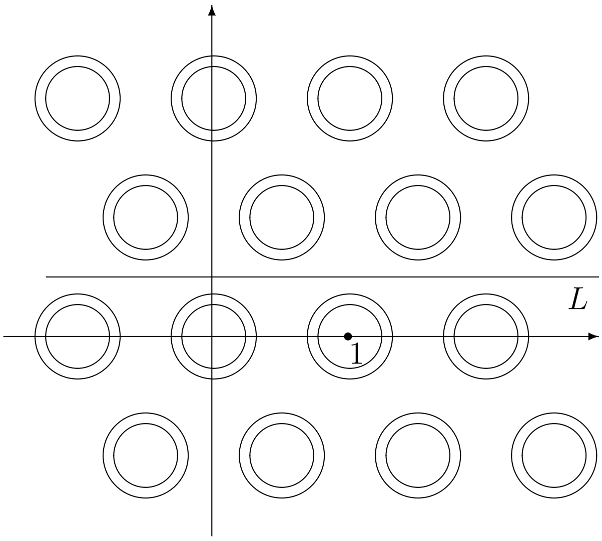
(a) Show that $\psi(H) \leq SO_2$. ***(4 marks)***

(b) Show that any two nontrivial rotations in H have the same centre. ***(5 marks)***

(c) Show that there is a point $a \in \mathbf{R}^2$ such that $h(a) = a$ for all $h \in H$. ***(3 marks)***

2 (i) For any subgroup $H \leq \mathrm{Isom}_2$ we defined its point group $\psi(H) \leq O_2$ and translation subgroup $\mathrm{Trans}(H) \leq \mathbf{R}^2$. Explain which properties $\psi(H)$ and $\mathrm{Trans}(H)$ need to satisfy for H to be a wallpaper group. ***(3 marks)***

(ii) Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (The axes and line marked L do not form part of the pattern. A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(a) Describe geometrically *all* the translations, reflections and rotations (if any) in G. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. ***(7 marks)***

(b) Find a list of four isometries that generate G. Justify your answer. ***(10 marks)***

(c) Let L be the line marked on the diagram, with equation $y = \sqrt{3}/4$. The glide reflection $G_{L,(1/2,0)}$ preserves the pattern. Write $G_{L,(1/2,0)}$ in terms of the generators you found in (b). ***(5 marks)***

3 (i) (a) Give the definition of the action of a group G on a set X. ***(3 marks)***

(b) Given a group action explain how to define the corresponding map $\phi : G \to S(X)$ and prove that it is a homomorphism taking values in $S(X)$. ***(5 marks)***

(ii) Let a group G act on itself by $g * x := xg^{-1}$.

(a) Show that this defines a group action. ***(2 marks)***

(b) Prove that $\mathrm{Orb}_G(x) = G$ for any $x \in G$. ***(2 marks)***

(c) Prove that the homomorphism $\phi : G \to S(G)$ corresponding to this action is injective. ***(2 marks)***

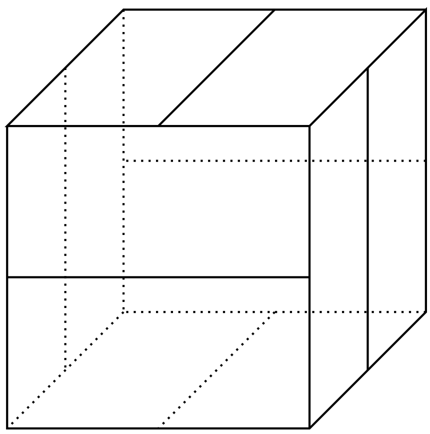
(d) Show that for any finite group of order $|G| > 2$ the homomorphism ϕ in (c) is not surjective. ***(2 marks)***

(iii) Let C be a cube centred at the origin in $\mathbf{R}^3$ and write

$$H = \mathrm{Dir}(C) = \{A \in \mathrm{SO}_3 | AC = C\}.$$

(a) Describe a set of four things on which H acts non-trivially. Let $\phi : H \to S_4$ the corresponding homomorphism. Describe a rotation that corresponds to the permutation (34) under this homomorphism. ***(3 marks)***

(b) We now add a stripe on each face of C as in the diagram below. Which elements of H send stripes to stripes? Which subgroup of S_4 do these rotations correspond to? ***(6 marks)***



4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) (a) Give the definition of a simple group. ***(2 marks)***

(b) By considering the action of G on the cosets of a Sylow 3-subgroup show that there is no simple group of order 45. ***(5 marks)***

(iii) Let G be a group of order 30.

(a) Let P be a Sylow 3-subgroup, Q a Sylow 5-subgroup. By considering the order of elements show that at least one of these is a normal subgroup of G. ***(4 marks)***

(b) Prove that $H = PQ$ is a subgroup of G. If $n_3 = n_5 = 1$ then prove that $H \lhd G$. ***(5 marks)***

(c) Now assume $n_2 = n_3 = n_5 = 1$. Prove that $G \cong R \times H$, where R denotes the Sylow 2-subgroup. ***(4 marks)***

End of Question Paper

Diagram for Question 2

Your registration number: ________________