SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester
2016–2017

MAS350 Measure and Probability

2 hours 30 minutes

Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Please leave this exam paper on your desk
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Registration number from U-Card (9 digits)
to be completed by student
(i) Given two probability measures $P_1$ and $P_2$ on a measurable space $(S, \Sigma)$ and a number $0 \leq c \leq 1$, define $P(A) = cP_1(A) + (1 - c)P_2(A)$ for all $A \in \Sigma$. Show that $P$ is also a probability measure on $(S, \Sigma)$. (4 marks)

(ii) (a) Let $(S, \Sigma, m)$ be a measure space and let $A_n, n \geq 1$ be an increasing sequence of subsets of $S$, that is $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Define $A := \bigcup_{n=1}^{\infty} A_n$. Show that

$$m(A) = \lim_{n \to \infty} m(A_n).$$

(HINT: Write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union.) (4 marks)

(b) Let $(S, \Sigma, m)$ be a measure space with $m$ a finite measure, that is $m(S) < \infty$. Let $B_n, n \geq 1$ be a decreasing sequence of subsets of $S$, that is $B_{n+1} \subseteq B_n$ for all $n \geq 1$. Define $B := \bigcap_{n=1}^{\infty} B_n$. Show that

$$m(B) = \lim_{n \to \infty} m(B_n).$$

(HINT: Consider $A_n = S - B_n$ and use part (a).) (5 marks)

(c) Give a counterexample to show that the above identity does not hold in general when the sets $B_n$ are decreasing and $m$ is an infinite measure. (2 marks)

(iii) Consider the set $S = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Write down the smallest $\sigma$-algebra $\Sigma$ of $S$ which contains $A$ and $B$. (5 marks)

(iv) Recall that the Borel $\sigma$-algebra $\mathcal{B} (\mathbb{R})$ is the smallest $\sigma$-algebra of $\mathbb{R}$ that contains open intervals $(a, b), -\infty \leq a < b \leq \infty$. Show that $\mathcal{B} (\mathbb{R})$ contains sets of the form $[a, b]$ and $\{a\}$, where $-\infty < a < b < \infty$. (5 marks)
(i) Consider the sequence \(a_n, \ n \geq 1\) given by
\[
a_n = \begin{cases} 
1 - \frac{1}{n}, & \text{if } n \text{ is odd} \\
-1 + \frac{1}{n}, & \text{if } n \text{ is even}
\end{cases}
\]
(a) Compute \(\limsup_{n \to \infty} a_n\) and \(\liminf_{n \to \infty} a_n\). \(\text{(3 marks)}\)
(b) Does \(\lim_{n \to \infty} a_n a_{n+1}\) exist? If so, what is the limit? \(\text{(2 marks)}\)

(ii) Let \((S, \Sigma)\) be a measurable space and let \(f, g : S \to \mathbb{R}\) be two measurable functions.
(a) Show that \(f + 1\) is a measurable function, where \(1 : S \to \mathbb{R}\) is the function which is identically one, \(1(s) = 1\) for all \(s \in S\). \(\text{(3 marks)}\)
(b) Show that \(\{f > g\} \in \Sigma\).
(HINT: If \(f(x) > g(x)\) then there must be a rational point between \(f(x)\) and \(g(x)\)) \(\text{(4 marks)}\)

(iii) Let \(S\) be a set and let \(A, B \subseteq S\). Show
\[1_{A \cap B} = 1_A \cdot 1_B,\]
where \(1_C\) denotes the indicator function of a set \(C\). \(\text{(4 marks)}\)

(iv) Give an example of a measurable space \((S, \Sigma)\) and a function \(f : S \to \mathbb{R}\) such that \(|f|\) is measurable but \(f\) is not. \(\text{(4 marks)}\)

(v) Let \(S\) be a set with \(A \subseteq S\) and consider the \(\sigma\)-algebra \(\Sigma = \{\emptyset, A, A^c, S\}\). Show that a function \(f : S \to \mathbb{R}\) is measurable if and only if it is constant on \(A\) and also constant on \(A^c\).
(HINT: For one direction consider \(f^{-1}\{a\}\) for \(a \in \mathbb{R}\).) \(\text{(5 marks)}\)
(i) Consider the function
\[ f(x) = \begin{cases} 
1, & 0 \leq x < 1 \\
-2, & 1 \leq x < 2 \\
-3, & 2 \leq x < 4 \\
2, & 10 \leq x < 11 \\
0, & \text{otherwise} 
\end{cases} \]

(a) Compute \( f_+ \) and \( f_- \). \( (4 \text{ marks}) \)
(b) Compute \( \int \! f(x) \, dx \). \( (4 \text{ marks}) \)

(ii) Let \( (S, \Sigma, m) \) be a measure space and \( f : S \to \mathbb{R} \) be an integrable function.

(a) Suppose \( m(f \neq 0) = 0 \). Show that \( \int_S f \, dm = 0 \).
(HINT: Write \( f = f \cdot 1_{\{f=0\}} + f \cdot 1_{\{f\neq0\}} \)) \( (4 \text{ marks}) \)
(b) Compute \( \int_0^1 1_{\mathbb{Q}}(x) \, dx \). \( (2 \text{ marks}) \)
(c) Suppose \( \int_S f \, dm = 0 \). Is it true that \( m(f \neq 0) = 0 \)? Prove it if it is true or else provide a counterexample if it is false. \( (3 \text{ marks}) \)

(iii) Let \( (S, \Sigma, m) \) be a measure space and let \( f, g : S \to \mathbb{R} \) be nonnegative measurable functions. Further suppose that \( f(x) \leq g(x) \) for all \( x \in S \). Show that
\[ \int_S f(x) \, m(dx) \leq \int_S g(x) \, m(dx). \]
(HINT: Use the definition of integral of a nonnegative measurable function) \( (4 \text{ marks}) \)

(iv) Consider \( ([0, 2], \mathcal{B}([0, 2]), \lambda) \) where \( \lambda \) is the Lebesgue measure on \( [0, 2] \). Let \( f : [0, 2] \to \mathbb{R} \) be a nonnegative integrable function such that \( f(x) \leq 3 \) for \( x \leq 1 \) and \( \int_1^2 f(x) \, dx = 2 \). Show that \( \lambda(x : f(x) \geq 4) \leq \frac{1}{2} \). \( (4 \text{ marks}) \)
(i) (a) State Fatou’s lemma. \( (3 \text{ marks}) \)

(b) Let \((S, \Sigma, m)\) be a measure space. Let \((f_n)\) be a sequence of non-negative measurable functions for which \(f_n \leq f\) for all \(n \in \mathbb{N}\), where \(f\) is integrable. Prove that

\[
\limsup_{n \to \infty} \int_S f_n \, dm \leq \int_S \limsup_{n \to \infty} f_n \, dm.
\]

(HINT: Apply Fatou’s lemma to \(f - f_n\).) \( (5 \text{ marks}) \)

(ii) (a) Consider the measure space \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)\) where \(\lambda\) is the Lebesgue measure. Show that if \(f : \mathbb{R} \to \mathbb{R}\) is integrable then so is the mapping \(x \to [1 - e^{-|x|}] f(x)\). \( (2 \text{ marks}) \)

(b) State the Dominated Convergence theorem and use it to find the limit

\[
\lim_{n \to \infty} \int_{\mathbb{R}} [1 - e^{-|x|/n}] \cdot e^{-|x|} \, dx.
\]

(iii) Let \((S, \Sigma, m)\) be a measure space. The Monotone Convergence theorem states that for any monotonic increasing sequence of non-negative measurable functions \(f_n\) from \(S\) to \(\mathbb{R}\) we have

\[
\int_S \lim_{n \to \infty} f_n \, dm = \lim_{n \to \infty} \int_S f_n \, dm.
\]

Give an example to show that the above identity does not hold if the functions \(f_n\) are monotonic decreasing. \( (3 \text{ marks}) \)

(iv) Let \((S, \Sigma, m)\) be a measure space.

(a) For integrable functions \(f, g\) show that

\[
\int_S |f + g| \, dm \leq \int_S |f| \, dm + \int_S |g| \, dm.
\]

(HINT: Triangle inequality) \( (3 \text{ marks}) \)

(b) We say that a sequence \((f_n, n \geq 1)\) of integrable functions from \(S\) to \(\mathbb{R}\) converges in \(L_1\) to an integrable function \(f\) if

\[
\lim_{n \to \infty} \int_S |f_n - f| \, dm = 0.
\]

Show that if \((g_n, n \geq 1)\) also converges in \(L_1\) to \(g\) then \(f_n + g_n\) converges to \(f + g\). \( (3 \text{ marks}) \)
(i) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X$ be a random variable that takes positive integer values.

(a) Deduce that $X = \sum_{i=1}^{\infty} 1_{\{X \geq i\}}$. 
(HINT: Consider the event $\{X(\omega) = k\}$) \hspace{1cm} (4 marks)

(b) Show that $\mathbb{E}(X) = \sum_{i=1}^{\infty} P(X \geq i)$. \hspace{1cm} (4 marks)

(ii) Prove that in any infinite sequence of independent (fair) coin tosses, the pattern $HTHHT$ appears infinitely often, where $H$ represents heads and $T$ represents tails. \hspace{1cm} (5 marks)

(iii) Let $X \sim \text{Poisson}(2)$, that is

$$P(X = k) = e^{-2} \cdot \frac{2^k}{k!}, \ k = 0, 1, \ldots$$

Compute the characteristic function of $X$. \hspace{1cm} (4 marks)

(iv) Let $X$ and $Y$ be independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and let $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions. Deduce that $f(X)$ and $g(Y)$ are also independent. \hspace{1cm} (4 marks)

(v) Consider the probability space $([0, 1], \mathcal{B}([0, 1]), \lambda)$ where $\lambda$ is the uniform measure on $[0, 1]$. Show that the random variables $X_n = n \cdot 1_{(0,n^{-1})}$ converge almost surely to $X \equiv 0$ but $X_n$ does not converge in mean square to $X$. \hspace{1cm} (4 marks)

End of Question Paper