The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS350 Measure and Probability **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Given two *probability* measures P_1 and P_2 on a measurable space (S, Σ) and a number $0 \le c \le 1$, define $P(A) = cP_1(A) + (1-c)P_2(A)$ for all $A \in \Sigma$. Show that P is also a probability measure on (S, Σ). ***(4 marks)***

(ii) (a) Let (S, Σ, m) be a measure space and let $A_n, n \ge 1$ be an increasing sequence of subsets of S, that is $A_n \subseteq A_{n+1}$ for all $n \ge 1$. Define $A := \cup_{n=1}^{\infty} A_n$. Show that

$$m(A) = \lim_{n \to \infty} m(A_n).$$

(HINT: Write $\cup_{n=1}^{\infty} A_n$ as a disjoint union.) ***(4 marks)***

(b) Let (S, Σ, m) be a measure space with m a finite measure, that is $m(S) < \infty$. Let $B_n, n \ge 1$ be a decreasing sequence of subsets of S, that is $B_{n+1} \subseteq B_n$ for all $n \ge 1$. Define $B := \cap_{n=1}^{\infty} B_n$. Show that

$$m(B) = \lim_{n \to \infty} m(B_n).$$

(HINT: Consider $A_n = S - B_n$ and use part (a).) ***(5 marks)***

(c) Give a counterexample to show that the above identity does not hold in general when the sets B_n are decreasing and m is an *infinite* measure. ***(2 marks)***

(iii) Consider the set $S = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Write down the smallest σ-algebra Σ of S which contains A and B. ***(5 marks)***

(iv) Recall that the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra of $\mathbb{R}$ that contains open intervals (a, b), $-\infty \le a < b \le \infty$. Show that $\mathcal{B}(\mathbb{R})$ contains sets of the form $[a, b]$ and $\{a\}$, where $-\infty < a < b < \infty$. ***(5 marks)***

2 (i) Consider the sequence a_n, $n \geq 1$ given by

$$a_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ is odd} \\ -1 + \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

(a) Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. ***(3 marks)***

(b) Does $\lim_{n\to\infty} a_n a_{n+1}$ exist? If so, what is the limit? ***(2 marks)***

(ii) Let (S, Σ) be a measurable space and let $f, g : S \to \mathbb{R}$ be two measurable functions.

(a) Show that $f + \mathbf{1}$ is a measurable function, where $\mathbf{1} : S \to \mathbb{R}$ is the function which is identically one, $\mathbf{1}(s) = 1$ for all $s \in S$. ***(3 marks)***

(b) Show that $\{f > g\} \in \Sigma$.
(HINT: If $f(x) > g(x)$ then there must be a rational point between $f(x)$ and $g(x)$) ***(4 marks)***

(iii) Let S be a set and let $A, B \subseteq S$. Show

$$\mathbf{1}_{A\cap B} = \mathbf{1}_A \cdot \mathbf{1}_B,$$

where $\mathbf{1}_C$ denotes the indicator function of a set C. ***(4 marks)***

(iv) Give an example of a measurable space (S, Σ) and a function $f : S \to \mathbb{R}$ such that $|f|$ is measurable but f is not. ***(4 marks)***

(v) Let S be a set with $A \subseteq S$ and consider the σ-algebra $\Sigma = \{\emptyset, A, A^c, S\}$. Show that a function $f : S \to \mathbb{R}$ is measurable if and only if it is constant on A and also constant on A^c.
(HINT: For one direction consider $f^{-1}(\{a\})$ for $a \in \mathbb{R}$.) ***(5 marks)***

3 (i) Consider the function

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ -2, & 1 \le x < 2, \\ -3, & 2 \le x < 4 \\ 2, & 10 \le x < 11, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute f_+ and f_-. ***(4 marks)***

(b) Compute $\int_{\mathbb{R}} f(x)\,dx$. ***(4 marks)***

(ii) Let (S, Σ, m) be a measure space and $f : S \to \mathbb{R}$ be an integrable function.

(a) Suppose $m(f \neq 0) = 0$. Show that $\int_S f dm = 0$.
(HINT: Write $f = f \cdot \mathbf{1}_{\{f=0\}} + f \cdot \mathbf{1}_{\{f\neq 0\}}$) ***(4 marks)***

(b) Compute $\int_0^1 \mathbf{1}_{\mathbb{Q}}(x)\,dx$. ***(2 marks)***

(c) Suppose $\int_S f dm = 0$. Is it true that $m(f \neq 0) = 0$? Prove it if it is true or else provide a counterexample if it is false. ***(3 marks)***

(iii) Let (S, Σ, m) be a measure space and let $f, g : S \to \mathbb{R}$ be nonnegative measurable functions. Further suppose that $f(x) \le g(x)$ for all $x \in S$. Show that

$$\int_S f(x)\,m(dx) \le \int_S g(x)\,m(dx).$$

(HINT: Use the definition of integral of a nonnegative measurable function) ***(4 marks)***

(iv) Consider $([0,2], \mathcal{B}([0,2]), \lambda)$ where λ is the Lebesgue measure on $[0,2]$. Let $f : [0,2] \to \mathbb{R}$ be a nonnegative integrable function such that $f(x) \le 3$ for $x \le 1$ and $\int_1^2 f(x)\,dx = 2$. Show that ***(4 marks)***

$$\lambda\left(x : f(x) \ge 4\right) \le \frac{1}{2}.$$

4 (i) (a) State Fatou's lemma. ***(3 marks)***

(b) Let (S, Σ, m) be a measure space. Let (f_n) be a sequence of non-negative measurable functions for which $f_n \leq f$ for all $n \in \mathbb{N}$, where f is integrable. Prove that

$$\limsup_{n\to\infty} \int_S f_n \, dm \leq \int_S \limsup_{n\to\infty} f_n \, dm.$$

(HINT: Apply Fatou's lemma to $f - f_n$.) ***(5 marks)***

(ii) (a) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue measure. Show that if $f : \mathbb{R} \to \mathbb{R}$ is integrable then so is the mapping $x \to \left[1 - e^{-|x|}\right] f(x)$. ***(2 marks)***

(b) State the Dominated Convergence theorem and use it to find the limit ***(6 marks)***

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left[1 - e^{-|x|/n}\right] \cdot e^{-|x|} \, dx.$$

(iii) Let (S, Σ, m) be a measure space. The *Monotone Convergence theorem* states that for any monotonic increasing sequence of non negative measurable functions f_n from S to $\mathbb{R}$ we have

$$\int_S \lim_{n\to\infty} f_n \, dm = \lim_{n\to\infty} \int_S f_n \, dm.$$

Give an example to show that the above identity does not hold if the functions f_n are monotonic decreasing. ***(3 marks)***

(iv) Let (S, Σ, m) be a measure space.

(a) For integrable functions f, g show that

$$\int_S |f + g| \, dm \leq \int_S |f| \, dm + \int_S |g| \, dm.$$

(HINT: Triangle inequality) ***(3 marks)***

(b) We say that a sequence $(f_n, n \geq 1)$ of integrable functions from S to $\mathbb{R}$ *converges in* $\mathcal{L}_1$ to an integrable function f if

$$\lim_{n\to\infty} \int_S |f_n - f| \, dm = 0.$$

Show that if $(g_n, n \geq 1)$ also converges in $\mathcal{L}_1$ to g then $f_n + g_n$ converges to $f + g$. ***(3 marks)***

5 (i) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable that takes positive *integer* values.

(a) Deduce that $X = \sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}}$.
(HINT: Consider the event $\{X(\omega) = k\}$) ***(4 marks)***

(b) Show that $\mathbb{E}(X) = \sum_{i=1}^{\infty} P(X \geq i)$. ***(4 marks)***

(ii) Prove that in any infinite sequence of independent (fair) coin tosses, the pattern $HTHHT$ appears infinitely often, where H represents heads and T represents tails. ***(5 marks)***

(iii) Let $X \sim \text{Poisson}(2)$, that is

$$P(X = k) = e^{-2} \cdot \frac{2^k}{k!},\ k = 0, 1, \cdots .$$

Compute the characteristic function of X. ***(4 marks)***

(iv) Let X and Y be independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and let $f, g : \mathbb{R} \to \mathbb{R}$ be Borel measurable functions. Deduce that $f(X)$ and $g(Y)$ are also independent. ***(4 marks)***

(v) Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where λ is the uniform measure on $[0,1]$. Show that the random variables $X_n = n \cdot \mathbf{1}_{(0,n^{-1})}$ converge almost surely to $X \equiv 0$ but X_n does not converge in mean square to X. ***(4 marks)***

End of Question Paper