Data provided: formula sheet

MAS352

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Stochastic Processes and Financial Mathematics **3 hours**

*Candidates should attempt **ALL** questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Let $\Omega = \{TT, HT, TH, HH\}$, representing the set of possible outcomes of two coin tosses, each of which may show either H (head) or T (tail). Let $X : \Omega \to \mathbb{R}$ be the total number of heads obtained.

(a) Write down the pre-images $X^{-1}(0)$, $X^{-1}(1)$ and $X^{-1}(2)$. ***(3 marks)***

(b) Write down all the elements of $\sigma(X)$. ***(6 marks)***

(c) Let

$$Y = \begin{cases} 1 & \text{if the first toss shows a tail,} \\ 0 & \text{otherwise.} \end{cases}$$

Is Y measurable with respect to $\sigma(X)$? Justify your answer. ***(2 marks)***

2 Let X be a random variable, where $|X| < 1$. Explain why

$$Y = \sum_{n=0}^{\infty} X^n$$

is a random variable. ***(4 marks)***

You may use standard results about measurability of sums, products and limits of random variables, providing they are clearly stated.

3 Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables, with the distribution of X_n given by $\mathbb{P}[X_n = n] = \frac{1}{n}$ and $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n}$.

(a) Show that $X_n \to 0$ in probability. ***(2 marks)***

(b) Show that X_n does not converge to zero in L^1. ***(2 marks)***

(c) Does X_n converge to a limit in L^1? Justify your answer briefly. ***(3 marks)***

4 Consider the one-period model, in discrete time, with two assets, cash and stock. Recall that, in the one-period model:

- If we hold x cash at time 0, it becomes worth $x(1+r)$ at time 1.
- If we hold y units of stock, worth yS_0 at time 0, it becomes worth yS_1 at time 1.

Here, $S_0 = s > 0$ is a deterministic constant, $0 < d < 1+r < u$ are also deterministic constants, and S_1 is a random variable with $\mathbb{P}[S_1 = su] = p_u$ and $\mathbb{P}[S_1 = sd] = p_d$, where $p_u + p_d = 1$ and $p_d, p_u \in (0,1)$.

(a) At time $t = 0$ our portfolio contains 2 units of cash and 5 units of stock. What is the value of this portfolio at time 1? ***(2 marks)***

(b) A rival investor holds a portfolio consisting of 3 units of cash and 4 units of stock. Under what condition, on d, r and u, can we be certain (at time 0) that our own portfolio will have strictly greater value at time 1? ***(3 marks)***

(c) Consider the contingent claim

$$\Phi(S_1) = \begin{cases} 2 & \text{if } S_1 = su, \\ -1 & \text{if } S_1 = sd. \end{cases}$$

Find a portfolio $h = (x, y)$, containing x cash and y units of stock, which replicates this contingent claim. ***(4 marks)***

(d) What is the arbitrage free price, at time 0, of the contingent claim $\Phi(S_1)$ in (c)? ***(2 marks)***

5 Consider the one-period model, as in Question 4. Let $K, M \in \mathbb{R}$ with $0 < K \leq M$. A contract states that

> *At time* 1*, the holder of this contract has the option to buy a single unit of stock, for a price* K*, provided that the current value of the stock is strictly less than* M*.*

Write down the contingent claim corresponding to this contract, and justify your answer. ***(4 marks)***

6 Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration, in discrete time.

(a) State the definition of a (discrete time) martingale M_n, with respect to $\mathcal{F}_n$. ***(4 marks)***

(b) Show that, if M_n is a martingale with respect to $\mathcal{F}_n$, then $\mathbb{E}[M_n] = \mathbb{E}[M_1]$ for all $n \in \mathbb{N}$. ***(3 marks)***

(c) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables, with common distribution

$$\mathbb{P}[X_n = 1] = \frac{2}{3}, \quad \mathbb{P}[X_n = -1] = \frac{1}{3}.$$

Define

$$S_n = \sum_{i=1}^{n} X_i$$

and suppose that $\mathcal{F}_n = \sigma(S_i \;:\; i = 1, 2, \ldots, n)$. Which of the following processes are martingales with respect to $\mathcal{F}_n$? Justify your answer in each case.

(i) $A_n = S_n$

(ii) $B_n = S_n - \frac{n}{3}$

(7 marks)

(d) Let M_n be a supermartingale with respect to $\mathcal{F}_n$, and suppose that for all $n \in \mathbb{N}$ we have $\mathbb{E}[M_n] = \mathbb{E}[M_1]$. Show that M_n is a martingale. ***(4 marks)***

7 Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion.

(a) Let $0 \le u \le t$. Write down the distribution of $B_t - B_u$. ***(2 marks)***

(b) Write down $\mathbb{E}[B_t]$ and $\mathbb{E}[B_t^2]$ as functions of t. ***(2 marks)***

(c) Let $0 \le u \le t$. Show that $\mathbb{E}[B_t B_u] = u$. ***(3 marks)***

8 Let $(B_u)_{u\in[0,\infty)}$ be a standard Brownian motion and let $T > 0$. Let X_u be an Ito process with stochastic differential

$$dX_u = 2X_u\,du + u\,dB_u. \tag{$\star$}$$

For $u \geq t$, let $\mathbb{E}_{t,x}[X_u]$ denote the expectation of X_u given the condition $X_t = x$, where $x \in \mathbb{R}$.

(a) Write ($\star$) in integral form over the time interval $[t, T]$. ***(2 marks)***

(b) For $u \geq t$, set $y(u) = \mathbb{E}_{t,x}[X_u]$. Show that

$$y(T) = x + \int_t^T 2y(u)\,du.$$

and write down a differential equation satisfied by $y(u)$. ***(5 marks)***

(c) For $T \geq t$, show that $\mathbb{E}_{t,x}[X_T] = xe^{2(T-t)}$. ***(2 marks)***

9 Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion. Let $\alpha \in \mathbb{R}$ and $\sigma > 0$. Let X_t be a geometric Brownian motion, satisfying the stochastic differential equation

$$dX_t = \alpha X_t\,dt + \sigma X_t\,dB_t.$$

with initial value $X_0 = 1$.

(a) Let $M_t = e^{-\alpha t}X_t$. Find the stochastic differential dM_t and hence show that M_t is a martingale. ***(7 marks)***

(b) Let $Y_t = X_t^2$. Find the stochastic differential dY_t and hence show that

$$Y_T = Y_t \exp\left\{\left(2\alpha - \sigma^2\right)(T-t) + 2\sigma(B_T - B_t)\right\}$$

for $t \in [0, T]$. ***(8 marks)***

(c) In the Black-Scholes model, with stock price S_t, show that the arbitrage free price at time $t \in [0, T]$ of a contract that has exercise date T and contingent claim $\Phi(S_T) = S_T^2$ is

$$S_t^2 \exp\left\{(r+\sigma^2)(T-t)\right\}.$$

(6 marks)

Standard notation, including the parameters r, μ and σ, and pricing formulae relating to the Black-Scholes model can be found on the supplementary sheet.

10 Within the Black-Scholes model, write down constant portfolios, to be bought at time 0, consisting of (any subset of) cash, stock, European call options and European put options, that replicate the following contingent claims:

(a) $\Phi_1(S_T) = 1 + 2S_T$. ***(2 marks)***

(b) $\Phi_2(S_T) = |S_T - K|$, where K is a (deterministic) constant. ***(2 marks)***

You should specify clearly the strike prices of any options that you choose to include in your portfolio.

(c) Is it possible to replicate the contingent claim $\Phi(S_T) = S_T^2$ with a constant portfolio consisting only of cash, stock, European call options and European put options? Justify your answer. ***(4 marks)***

End of Question Paper

MAS352/452/6052 – Formula Sheet

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t,x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t)\,dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$, with initial value S_0.

At time $t \in [0, T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\begin{aligned} \frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) &= 0, \\ F(T,s) &= \Phi(s). \end{aligned}$$

The unique solution F satisfies

$$F(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\Phi(S_T) \mid \mathcal{F}_t]$$

for all $t \in [0, T]$. Here, the 'risk-neutral world' $\mathbb{Q}$ is the probability measure under which S_t satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$