The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Time Series **2 hours**

Marks will be awarded for your best ***three*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 60 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Explain briefly why the definition of the sample autocorrelation function is not appropriate for non-stationary time series. ***(2 marks)***

(ii) (a) For the following time series data

time (t)	response (y_t)
1	1
2	3
3	4
4	10
5	2
6	2
7	5
8	11

calculate 4-span moving averages for time points $t = 3, 4, 5, 6$. ***(4 marks)***

(b) Consider the time series

$$y_t = \alpha + \beta t,$$

for some known α and β and for $t = 1, 2, 3, \ldots$. Show that the 4-span moving average of $\{y_t\}$ at time t is exactly equal to the value of y_t. ***(4 marks)***

(c) Now consider the time series

$$x_t = y_t + \epsilon_t,$$

where y_t is the time series in part (b) above and ϵ_t is white noise with variance 2. Show that the de-trended time series defined by

$$d_t = x_t - \hat{y}_t,$$

where $\hat{y}_t$ is the 4-span moving average of y_t, is a weakly stationary time series. ***(2 marks)***

(iii) Let $\{Z_t\}$ be a sequence of independent identically distributed random variables, so that Z_t follows a normal distribution with zero mean and variance 1.

Define the process

$$y_t = \begin{cases} Z_t, & \text{if } t \text{ is even} \\ \dfrac{Z_t^2 - 1}{\sqrt{2}}, & \text{if } t \text{ is odd.} \end{cases}$$

(a) Show that $\{y_t\}$ is a white noise process $WN(0, 1)$. ***(6 marks)***

(b) Show $\{y_t\}$ is not an identically distributed sequence. ***(2 marks)***

HINT: If a random variable X follows the chi-square distribution with ν degrees of freedom $X \sim \chi^2_\nu$, then $E(X) = \nu$ and $\text{Var}(X) = 2\nu$.

2 Consider the autoregressive time series model

$$y_t = c + \frac{1}{3}y_{t-1} - \frac{1}{2}y_{t-2} + \epsilon_t, \quad (1)$$

where ϵ_t is white noise with variance 1.

(i) Write model (1) in compact form using the backward shift operator B. ***(1 mark)***

(ii) Show that model (1) is causal. ***(3 marks)***

(iii) If the mean of y_t is equal to 0.5, then calculate c. ***(2 marks)***

(iv) Calculate the variance of y_t. ***(6 marks)***

(v) A quarterly time series $\{x_t\}$ is modelled so that $x_t - x_{t-4} = y_t$ follows model (1). The first 8 observations are collected and are shown in the table below

time (t)	response (x_t)
1	1
2	2
3	2
4	10
5	0
6	1
7	4
8	13

(a) Compute the one-step forecast mean of x_9 given observations $x_1, \ldots, x_8$ and the two-step forecast mean of x_{10}, given observations $x_1, \ldots, x_8$. ***(4 marks)***

(b) Compute a 95% one-step forecast interval for the observation x_9. ***(4 marks)***

3 Suppose that observations $y_1, y_2, \ldots, y_n$ are generated from the autoregressive (AR) model of order one:

$$y_t = \alpha y_{t-1} + \epsilon_t,$$

where α is the AR parameter and ϵ_t is a Gaussian white noise with variance σ^2.

(i) Write down the likelihood function $L(\alpha, \sigma^2; y_{1:n})$ and the log-likelihood function $\ell(\alpha, \sigma^2; y_{1:n})$ of the parameters α and σ^2, based on observation $y_{1:n} = \{y_1, y_2, \ldots, y_n\}$. ***(3 marks)***

(ii) Using the approximation $\log(1 + x) \approx x$, for some x, with $|x| < 1$, show that the log-likelihood of part (i) can be approximated as

$$\ell(\alpha, \sigma^2; y_{1:n}) \approx -\frac{n}{2}\log(2\pi\sigma^2) - \frac{y_1^2}{2\sigma^2} + \left(\frac{y_1^2 - \sigma^2}{2\sigma^2}\right)\alpha^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{n}(y_t - \alpha y_{t-1})^2.$$

(7 marks)

(iii) Using (ii) and adopting unconditional least squares, show that the approximate likelihood estimates of α and σ^2 satisfy

$$\hat{\alpha} = \frac{\sum_{t=2}^{n} y_t y_{t-1}}{\sum_{t=3}^{n} y_{t-1}^2 + \hat{\sigma}^2} \quad \text{and} \quad \hat{\sigma}^2 = \frac{(1 - \hat{\alpha}^2)y_1^2 + \sum_{t=2}^{n}(y_t - \hat{\alpha} y_{t-1})^2}{n}.$$

(8 marks)

(iv) If $\sigma^2 = y_1^2$, show that the maximum likelihood of α based on unconditional least squares is approximately equal to the maximum likelihood of α using conditional least squares. ***(2 marks)***

4 An airliner on a long-haul journey is set to fly at a pre-specified steady speed of v km/hr, but atmospheric conditions sometimes speed it up and sometimes slow it down. A simple model for its true position x_t along the flight path at time t hours supposes that

$$x_{t+1} = x_t + v + \eta_t,$$

where $\{\eta_t\}$ is a zero mean normal white noise sequence with variance σ_η^2. The position x_t can be observed only with error, the observed position y_t at time t being

$$y_t = x_t + \epsilon_t,$$

where ϵ_t is a zero mean normal white noise with variance σ_ϵ^2, uncorrelated with the other variables.

(i) Show that if $\beta_t = (x_t, v)^\top$, the system can be described by the equations

$$y_t = g^\top \beta_t + \epsilon_t, \quad (2)$$

$$\beta_t = F\beta_{t-1} + \zeta_t, \quad (3)$$

where g is a constant vector, F is a constant matrix and ζ_t is a random vector. Give the values of g and F and write down the mean and covariance matrix of ζ_t. What are equations (2) and (3) called in the context of state space linear modelling? ***(4 marks)***

(ii) Suppose that hourly observations of the plane's position are available up to time $t-1$ and it is believed that the true position x_{t-1} at time $t-1$ has posterior normal distribution with mean m_{t-1} and variance v_{t-1}. Show that, given this information, the prior mean of the position x_t at time t is $m_{t-1} + v$ and that the prior variance is $v_{t-1} + \sigma_\eta^2$. ***(4 marks)***

(iii) Find the posterior distribution of the true position x_t at time t when an observation y_t of the plane's position at time t becomes available. ***(8 marks)***

(iv) Suppose that $v = 720$, $\sigma_\epsilon = 10$ and $\sigma_\eta = 30$. If, one hour into the flight, an observation gives the position as $y_1 = 750$ km and it was known with certainty that at time $t = 0$ the true position had been $x_0 = 0$, find the posterior mean of the plane's position at $t = 1$, and the associated posterior variance. Hence give an approximate 95% credible interval for the true position x_1 after one hour. ***(4 marks)***

End of Question Paper