Answer all four questions. Formulae are on the last page.

1. (i) Consider the equation in the form

\[ \nabla \left( p + \frac{B^2}{2\mu_0} \right) - \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = 0. \]

For a magnetic field \( \mathbf{B} = B_0 \tanh \frac{x}{L} \hat{y} \), where \( \hat{y} \) is the unit vector along the \( y \)-direction, in the Cartesian coordinate system \((x, y, z)\), show that

\[ p + \frac{B^2}{2\mu_0} \]

is constant. In the above, \( p \) is gas pressure, \( \mu_0 \) is the magnetic permeability in a vacuum and \( L \), a typical length scale. \( \text{(3 marks)} \)

(ii) In cylindrical coordinates \((\hat{r}, \hat{\theta}, \hat{z})\), consider a purely azimuthal magnetic field \( \mathbf{B} = \frac{B_0}{r} \hat{\theta} \), where \( B_0 \) is a constant. Calculate the current density, vector potential and magnetic pressure gradient. \( \text{(10 marks)} \)

(iii) A static radially symmetric corona with temperature

\[ T(r) = T_0 \left( \frac{r_0}{r} \right)^{2/7} \]

is in equilibrium under a balance between a pressure gradient and gravity, \( \left( \frac{MG \rho}{r^2} \right) \hat{r} \) where \( M \) is the mass and \( G \) is the universal gravitational constant. \( T_0 \), \( \rho_0 \) and \( \rho_0 \) are the temperature, pressure and density respectively at a reference distance \( r = r_0 \). Find the pressure \( p(r) \) and density \( \rho(r) \). Show that according to this model the pressure at large distances is much greater than the interstellar pressure of \( \rho_0/10^{15} \). Comment on this last fact.

[Hint: use \( \frac{MG}{r_0 R T_0} = 15 \); \( R \) is a gas constant.] \( \text{(12 marks)} \)
Consider a magnetic field
\[ \mathbf{B} = \left( \frac{\partial \psi}{\partial z}, B_y(x, z), - \frac{\partial \psi}{\partial x} \right), \]
where \( \psi = \psi(x, z) \).

(a) Show that \( \nabla \cdot \mathbf{B} = 0 \). \hfill (2 marks)

(b) Show that \( \mathbf{B} \cdot \nabla \psi = 0 \) and that projections of field lines in the \( xz \)-plane are given by \( \psi = \text{constant} \). \hfill (6 marks)

(c) Show that if the Lorentz force
\[ \mathbf{J} \times \mathbf{B} = 0, \]
then
\[ B_y = B_y(\psi) \]
and \( \psi \) satisfies
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + B_y \frac{dB_y}{d\psi} = 0. \]
\hfill (12 marks)

(ii) Calculate the approximate timescale (in years) for the decay of the interstellar magnetic field given the parameters: length scale \( L = 3 \times 10^{18} \) cm and magnetic diffusivity \( \eta = 3.6 \times 10^8 \) cm$^2$/s. \hfill (5 marks)

3 (i) Verify that a solution of the form \( B(x, t) = f(t)e^{-x^2/(4\eta t)} \) satisfies the diffusion equation
\[ \frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}, \]
if
\[ \frac{df}{dt} = -\frac{f}{2t}. \]
Here \( \eta \) is the magnetic diffusivity. Integrate to find \( f(t) \) and therefore \( B(x, t) \) if \( B(0, t_0) = B_0 \) [i.e., if \( f(t_0) = B_0 \)]. \hfill (13 marks)

(ii) Verify that the magnetic field \( \mathbf{B}(x, y, z) = (x, y, 6y - 2z) \) is solenoidal. Calculate the equations of the field lines in \( yz \)-plane. \hfill (8 marks)

(iii) Sketch the magnetic fields for \( \mathbf{B} = (x, 0, 1)e^{-z} \) on the \( xz \)-plane with arrows indicating the direction of the field. \hfill (4 marks)
4. (i) Write down the linearised equations of MHD for adiabatic perturbations about a uniform state at rest. [Hint: consider an inviscid, perfectly conducting, incompressible fluid with no gravity]. 

\[ \text{(5 marks)} \]

Deduce what they become for perturbations that are proportional to \( \exp[i(\omega t - k \cdot r)] \).

\[ \text{(5 marks)} \]

Derive the dispersion relation for Alfvén waves from these equations.

\[ \text{(5 marks)} \]

(ii) Find how the non-dimensional flow given by \( \mathbf{v} = (\sin z, \cos z, 0) \) in the Eulerian representation deforms the non-dimensional magnetic field given by \( \mathbf{B} = (0, 0,1) \) at \( t = 0 \).

\[ \text{(10 marks)} \]
Formulae Sheet

\[ \nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \]

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\[ \nabla \cdot \mathbf{V} = \frac{1}{fgh} \left[ \frac{\partial}{\partial u} (ghV_u) + \frac{\partial}{\partial v} (fhV_v) + \frac{\partial}{\partial w} (fgV_w) \right] \]

\[ \nabla \times \mathbf{V} = \frac{1}{gh} \left[ \frac{\partial}{\partial v} (hV_v) - \frac{\partial}{\partial w} (gV_v) \right] \hat{\mathbf{u}} + \frac{1}{fh} \left[ \frac{\partial}{\partial w} (fV_w) - \frac{\partial}{\partial u} (hV_w) \right] \hat{\mathbf{v}} + \frac{1}{fg} \left[ \frac{\partial}{\partial u} (gV_u) - \frac{\partial}{\partial v} (fV_v) \right] \hat{\mathbf{w}} \]

vector identity:

\[ \nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \]

End of Question Paper