The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2016-2017**

Advanced Operations Research **2 Hours**

Attempt all FOUR questions.

1 (i) We consider the following linear programming problem:

$$\max \;\; z = 2x_1 - x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 6, \qquad 2x_1 - 2x_2 \leq 1.$$

Introducing slack variables x_3 and x_4, we use the simplex method to solve the problem. After a few iterations, the following tableau is found:

Basis	x_1	x_2	x_3	x_4	Solution
z	0	-1	0	1	1
x_3	0	2	1	$-1/2$	$11/2$
x_1	1	-1	0	$1/2$	$1/2$

Is the tableau already optimal? Give your reasons. If it is not, use the simplex method to continue the calculation and find the optimal solution. ***(7 marks)***

(ii) We use the two-phase method to solve the following linear programming problem:

$$\max \;\; z = 3x_1 - 5x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + 2x_2 \leq 6, \quad 3x_1 + x_2 \geq 2, \quad x_1 - x_2 \leq 1.$$

Construct the initial tableau in phase 1 and preprocess it so that it is suitable to be solved using the simplex method. **Do NOT proceed further.** ***(8 marks)***

(iii) Alice, Bob, Carrol, and Dan are working as a team in a competition in which they are given 6 tasks, and a team wins by completing the most tasks. Each member can conduct only the tasks in his/her skill set. Some members are more experienced, hence can undertake more tasks than others. The skill set and the maximum number of tasks each member can undertake are summarized in the table below.

Member	Tasks within skill sets	Maximum number of tasks
Alice	$1, 3, 5, 6$	3
Bob	$2, 4, 5$	2
Carrol	$1, 4, 6$	1
Dan	$2, 3, 5, 6$	3

The team wants to formulate a maximal flow model to find the optimal task allocation, so that they can maximise the total number of tasks they may be able to finish. Sketch and annotate properly the network of the problem. **Do NOT attempt to find the mathematical formulation; do NOT attempt to find the solution of the model either**. ***(10 marks)***

2 (i) Chinaco makes three types of tableware: plates, cups, and bowls. The following information is given:

- A one time setup cost is required if a product is to be made. The setup cost for making the plates is £100. Cups and bowls are made by another machine that costs £80 to set up. Since cups and bowls are made by the same machine, the machine needs to be set up only once if both are to be produced.
- The tablewares can be made in two ways, either by using material A only, or by mixing material B and material C.

The table below provides the pertinent data regarding the use of raw materials, labour time, their availability, and the estimated unit profits.

	Plates	Cups	Bowls	Availability
Material A (grams/unit)	500	20	90	5000 grams
Material B (grams/unit)	200	10	45	3000 grams
Material C (grams/unit)	350	15	40	2200 grams
Labour time (hrs/unit)	10	3	5	250 hours
Profit per unit (£)	3	2	2	

Formulate the mixed integer linear programming model you may use to find the optimum number of units that Chinaco should manufacture of each product. **Find the formulation only; do NOT attempt to solve it**.

(14 marks)

(ii) As the owner of a small shop, you are making plans to purchase a product from a supplier, which you will sell to your customers.

- Based on market projections, it is assumed that you will sell 300, 320, 450, and 250 units of the product in the next four quarters, respectively.
- The price per unit to **purchase from the supplier** starts at £20 in the first quarter and increases by £2 each quarter thereafter.
- The supplier can provide no more than 400 units in any one quarter.
- Although we can take advantage of lower prices in early quarters, a storage cost of £7 is incurred per unit per quarter if the product needs to be held over from one quarter to the next. No storage cost is incurred if the product is purchased and sold in the same quarter.

Develop a linear programming model to minimize the total expense on purchase and storage. **Find the formulation only; do NOT attempt to solve it**.

(11 marks)

3 We consider the following general maximisation problem with both inequality constraints and equation constraints:

$$\max\ z = f(x), \quad \text{subject to} \quad g(x) \leq 0 \quad \text{and} \quad h(x) = 0, \tag{1}$$

where $x \in R^n$ and, in general, $g(x)$ and $h(x)$ are vector functions. You are given the following definitions.

- The Lagrangian function for this problem is defined as

$$L(x, y, w) = f(x) - y^T g(x) - w^T h(x) \tag{2}$$

  where y and w are the dual variables; $y \geq 0$ but w can **take any real values**.

- The corresponding Lagrangian dual function is defined by

$$v(y, w) = \max_{x \in R^n} L(x, y, w). \tag{3}$$

- The dual problem is defined as

$$\min\ v(y, w) \quad \text{subject to} \quad y \geq 0. \tag{4}$$

Based on this given information, answer the following questions.

(i) Let v_{min} be the minimum cost for the dual problem in Equation (4) and z_{max} be the maximum cost for the primal problem in Equation (1). Show that $v_{\text{min}} \geq z_{\text{max}}$. ***(8 marks)***

(ii) We now consider a maximisation LPP in its canonical form:

$$\max\ z = c^T x, \quad \text{subject to} \quad \overline{A}x = b,\ x \geq 0, \tag{5}$$

where x includes both the decision variables and the slack variables, and $\overline{A} = [A, I]$ according to the usual notation. Using the definitions given in Equations (2), (3) and (4), write down the Lagrangian function for the LPP, hence derive its dual problem. ***(9 marks)***

(iii) Without introducing the slack variables, the LPP in Equation (5) is given by

$$\max\ z = c^T x, \quad \text{subject to} \quad Ax \leq b,\ x \geq 0. \tag{6}$$

As is shown in lectures, the dual problem can be written as

$$\min\ v = b^T y, \quad \text{subject to} \quad A^T y \geq c,\ y \geq 0. \tag{7}$$

Show that the dual problem in Equation (7) is the same as the dual problem derived in Part (ii) of this question. ***(8 marks)***

4 A company produces four types of trolleys: T1, T2, T3, and T4. To make one trolley, certain amounts of machine time and worker-days are needed. Relevant information is given in the following table:

	T1	T2	T3	T4	Availability
Machine time (hours/unit)	1	3	8	4	90
Worker-days (per unit)	1	1	1	3	80
Profit (£/unit)	1	2	4	3	

To find the optimal production schedule, we let x_1, x_2, x_3 and x_4 be the numbers of trolleys T1, T2, T3 and T4 to be manufactured, and x_5 and x_6 be the slack variables corresponding to the machine time and worker-day constraints, respectively. The linear programming model is formulated as follows:

$$\begin{aligned} \max \quad & z = x_1 + 2x_2 + 4x_3 + 3x_4 \, , \\ \text{subject to} \quad & x_1 + 3x_2 + 8x_3 + 4x_4 \leq 90 \, , \\ & x_1 + x_2 + x_3 + 3x_4 \leq 80 \, . \end{aligned}$$

Using the simplex method, the optimal tableau has been determined to be:

	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	1/2	1/2	1/2	1/2	85
x_2	0	1	7/2	1/2	1/2	−1/2	5
x_1	1	0	−5/2	5/2	−1/2	3/2	75

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables, the cost function, and the optimal solutions for the dual variables. ***(3 marks)***

(ii) Using the information given by the above tableau, verify that the complementary slackness conditions are satisfied for both the above problem and its dual problem. ***(4 marks)***

(iii) Determine the range of the unit profit for trolley T1 which leaves the current optimal basis unchanged. ***(8 marks)***

(iv) The company is considering manufacturing one more type of trolley. To make one unit of this type of trolley requires 1 hour of machine time and 2 worker-days. The profit for this unit is £3. Determine whether the company should indeed manufacture this type of trolley. ***(4 marks)***

(v) Find the range of values for available worker-days for which the optimal basis remains the same. ***(3 marks)***

(vi) Thanks to technology innovation, it is now possible to make trolley T3 with only 4 hours of machine time and 2 worker-days. If this innovation is adopted, does the solution remain optimal? Give your reasons. ***(3 marks)***

End of Question Paper