1 (i) (a) What is a topological space? If $X$ is a topological space and $\sim$ is an equivalence relation on $X$, define the quotient topology on $Y := X/\sim$. 

(b) Suppose $X$ is a topological space, give $Y = X/\sim$ the quotient topology, and let $\pi : X \rightarrow Y$ be the quotient map. Prove that if $f : Y \rightarrow Z$ is a function, then $f$ is continuous if and only if $f \circ \pi$ is continuous. Prove that if $X$ is compact then $Y$ is compact. Does the reverse implication hold? 

(ii) (a) If $X$ is a topological space and $x_0 \in X$, what is a loop based at $x_0$? 

(b) Suppose that $\omega, \sigma$ are two loops based at $x_0$, define the reverse loop $\overline{\omega}$ and the concatenated loop $\omega \cdot \sigma$. Show that $\omega \cdot \overline{\omega}$ is loop-homotopic to the constant loop. 

(c) Show that if $\omega$ is any loop in $X$ based at $x_0$, then there is a continuous map $f : S^1 \rightarrow X$ so that $[\omega] \in \text{im}(f_* : \pi_1(S^1, 1) \rightarrow \pi_1(X, x_0))$. Deduce that if $\pi_1(X, x_0)$ is non-trivial for some topological space $X$, then so too is $\pi_1(S^1, 1)$. 

Turn Over
2 (i) (a) What is a covering map? (4 marks)

(b) State the Path Lifting Lemma for a covering map \( p : Y \rightarrow X \), and explain how it can be used to define a function

\[
\ell : \pi_1(X, x_0) \rightarrow p^{-1}(x_0),
\]

where \( x_0 \in X \). State conditions under which \( \ell \) is a bijection. (8 marks)

(ii) (a) Show that if \( Y \) is a Hausdorff space and \( y_1, \ldots, y_n \) are \( n \) distinct points then there are disjoint open sets \( V_1, \ldots, V_n \) with \( y_i \in V_i \). (2 marks)

(b) Suppose \( G \) is a finite group of order \( n \) acting freely on a Hausdorff space \( Y \) (i.e., for any \( y \in Y \) the orbit \( \{gy \mid g \in G\} \) of \( y \) has \( n \) elements). Show that if \( X = Y/G \) is the space of orbits with the quotient topology, then the quotient map \( q : Y \rightarrow X \) is a covering map. (6 marks)

(c) Assuming that the group \( SU(n) \) is simply connected and \( G \) is a finite subgroup of \( SU(n) \), construct a space with fundamental group \( G \). [The group \( SU(n) \) consists of \( n \times n \) complex matrices \( A \) with \( AA^T = I \) and \( \det(A) = 1 \), but all you need to know is that it is Hausdorff and that the group multiplication is continuous.] (5 marks)

3 (i) (a) What is a chain complex of abelian groups? What is the homology of such a chain complex? (4 marks)

(b) State the Mayer-Vietoris Theorem for calculating the homology of a simplicial complex \( K = L \cup M \) expressed as the union of two subcomplexes \( L \) and \( M \). (5 marks)

(ii) (a) If \( K \) is a simplicial complex and \( P \) is a new vertex, what is the \( P \)-cone \( c_PK \) on \( K \)? Show that for any \( K \), the homology of \( c_PK \) is the homology of a point. (10 marks)

(b) Suppose \( L \) is a geometric simplicial complex in \( \mathbb{R}^n \) and take the two points \( N = (0, \ldots, 0, +1) \), \( S = (0, \ldots, 0, -1) \) in \( \mathbb{R}^{n+1} \). Let \( \Sigma L = c_NL \cup c_SL \) be the union of the \( N \)-cone and the \( S \)-cone on \( L \), and show that

\[
H_{i+1}(\Sigma L) = H_i(L)
\]

for \( i \geq 1 \). (6 marks)
4 (i) (a) Suppose that $C_\bullet$ is a chain complex with only finitely many terms non-zero and all terms finite dimensional vector spaces over $\mathbb{Q}$. What is the Lefschetz number $\Lambda(\theta)$ of a chain map $\theta : C_\bullet \to C_\bullet$? (2 marks)

(b) Show that $\Lambda(\theta) = \Lambda(\theta_*)$ where $\theta_* : H_*(C_\bullet) \to H_*(C_\bullet)$ is the induced map in homology. (6 marks)

(c) State the Lefschetz Fixed Point Theorem. (2 marks)

(ii) Consider maps $f : M(g) \to M(g)$, where $M(g)$ denotes a compact orientable surface of genus $g \geq 0$.

(a) Write down the homology of $M(g)$. For which $g$ is there a self-map $f$ without fixed points so that $f \simeq id_{M(g)}$? (7 marks)

(b) Suppose that $f \simeq id$ and that $f$ is a simplicial isomorphism for some triangulation. Show that $f$ cannot have exactly one fixed point $P$. [Hint: Consider $f$ restricted to the punctured surface $M(g) \setminus \{P\}$.] (8 marks)

5 Are the following true or false. Justify your answers.

(i) Any self-map of the projective plane $\mathbb{R}P^2$ has a fixed point. (5 marks)

(ii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$. (5 marks)

(iii) There is a covering map $T^2 \to S^2$ from the 2-torus to the 2-sphere. (5 marks)

(iv) If $X$ is obtained from the 2-torus $T^2$ by deleting one point, then $X$ is homotopy equivalent to a 1-dimensional geometric simplicial complex. (5 marks)

(v) The space $X = S^2 \cup \mathbb{B}^2$ is homotopy equivalent to $S^2$. [Here $S^2$ is the unit sphere in $\mathbb{R}^3$ centred at the origin and $\mathbb{B}^2$ is the unit disc in the $(x, y)$-plane centred at the origin.] (5 marks)

End of Question Paper