The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2016–2017

MAS435 Algebraic Topology 2 hours 30 minutes

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) (a) What is a *topological space*? If X is a topological space and $\sim$ is an equivalence relation on X, define the *quotient topology* on $Y := X/\sim$. ***(5 marks)***

(b) Suppose X is a topological space, give $Y = X/\sim$ the quotient topology, and let $\pi : X \longrightarrow Y$ be the quotient map. Prove that if $f : Y \longrightarrow Z$ is a function, then f is continuous if and only if $f \circ \pi$ is continuous. Prove that if X is compact then Y is compact. Does the reverse implication hold? ***(7 marks)***

(ii) (a) If X is a topological space and $x_0 \in X$, what is a *loop* based at x_0? ***(2 marks)***

(b) Suppose that ω, σ are two loops based at x_0, define the *reverse* loop $\overline{\omega}$ and the *concatenated* loop $\omega \cdot \sigma$. Show that $\omega \cdot \overline{\omega}$ is loop-homotopic to the constant loop. ***(7 marks)***

(c) Show that if ω is any loop in X based at x_0, then there is a continuous map $f : S^1 \longrightarrow X$ so that $[\omega] \in \mathrm{im}(f_* : \pi_1(S^1, 1) \longrightarrow \pi_1(X, x_0))$. Deduce that if $\pi_1(X, x_0)$ is non-trivial for some topological space X, then so too is $\pi_1(S^1, 1)$. ***(4 marks)***

2 (i) (a) What is a *covering map*? ***(4 marks)***

(b) State the Path Lifting Lemma for a covering map $p : Y \longrightarrow X$, and explain how it can be used to define a function

$$\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0),$$

where $x_0 \in X$. State conditions under which ℓ is a bijection.
(8 marks)

(ii) (a) Show that if Y is a Hausdorff space and $y_1, \ldots, y_n$ are n distinct points then there are disjoint open sets $V_1, \ldots, V_n$ with $y_i \in V_i$.
(2 marks)

(b) Suppose G is a finite group of order n acting freely on a Hausdorff space Y (i.e., for any $y \in Y$ the orbit $\{gy \mid g \in G\}$ of y has n elements). Show that if $X = Y/G$ is the space of orbits with the quotient topology, then the quotient map $q : Y \longrightarrow X$ is a covering map. ***(6 marks)***

(c) Assuming that the group $SU(n)$ is simply connected and G is a finite subgroup of $SU(n)$, construct a space with fundamental group G. [The group $SU(n)$ consists of $n \times n$ complex matrices A with $A\overline{A}^t = I$ and $\det(A) = 1$, but all you need to know is that it is Hausdorff and that the group multiplication is continuous.]
(5 marks)

3 (i) (a) What is a *chain complex* of abelian groups? What is the *homology* of such a chain complex? ***(4 marks)***

(b) State the Mayer-Vietoris Theorem for calculating the homology of a simplicial complex $K = L \cup M$ expressed as the union of two subcomplexes L and M. ***(5 marks)***

(ii) (a) If K is a simplicial complex and P is a new vertex, what is the P-cone $c_P K$ on K? Show that for any K, the homology of $c_P K$ is the homology of a point. ***(10 marks)***

(b) Suppose L is a geometric simplicial complex in $\mathbb{R}^n$ and take the two points $N = (0, \ldots, 0, +1), S = (0, \ldots, 0, -1)$ in $\mathbb{R}^{n+1}$. Let $\Sigma L = c_N L \cup c_S L$ be the union of the N-cone and the S-cone on L, and show that

$$H_{i+1}(\Sigma L) = H_i(L)$$

for $i \geq 1$. ***(6 marks)***

4 (i) (a) Suppose that $C_\bullet$ is a chain complex with only finitely many terms non-zero and all terms finite dimensional vector spaces over $\mathbb{Q}$. What is the *Lefschetz number* $\Lambda(\theta)$ of a chain map $\theta : C_\bullet \longrightarrow C_\bullet$? ***(2 marks)***

(b) Show that $\Lambda(\theta) = \Lambda(\theta_*)$ where $\theta_* : H_*(C_\bullet) \longrightarrow H_*(C_\bullet)$ is the induced map in homology. ***(6 marks)***

(c) State the Lefschetz Fixed Point Theorem. ***(2 marks)***

(ii) Consider maps $f : M(g) \longrightarrow M(g)$, where $M(g)$ denotes a compact orientable surface of genus $g \geq 0$.

(a) Write down the homology of $M(g)$. For which g is there a self-map f without fixed points so that $f \simeq id_{M(g)}$? ***(7 marks)***

(b) Suppose that $f \simeq id$ and that f is a simplicial isomorphism for some triangulation. Show that f cannot have exactly one fixed point P. [Hint: Consider f restricted to the punctured surface $M(g) \setminus \{P\}$.] ***(8 marks)***

5 Are the following true or false. Justify your answers.

(i) Any self-map of the projective plane $\mathbb{R}P^2$ has a fixed point. ***(5 marks)***

(ii) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$. ***(5 marks)***

(iii) There is a covering map $T^2 \longrightarrow S^2$ from the 2-torus to the 2-sphere. ***(5 marks)***

(iv) If X is obtained from the 2-torus T^2 by deleting one point, then X is homotopy equalent to a 1-dimensional geometric simplicial complex. ***(5 marks)***

(v) The space $X = S^2 \cup \overline{B}^2$ is homotopy equivalent to S^2. [Here S^2 is the unit sphere in $\mathbb{R}^3$ centred at the origin and $\overline{B}^2$ is the unit disc in the (x, y)-plane centred at the origin.] ***(5 marks)***

End of Question Paper