Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form \[
\begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
\]
The standard symplectic form \(\Omega\) on \(\mathbb{R}^{2n}\) is defined by \(\Omega(Z, Z') = Q \cdot P' - P \cdot Q'\), where \(Z = (Q, P)\) and \(Z' = (Q', P')\) are elements of \(\mathbb{R}^{2n}\).

You may use without proof the fact that \(\det S = 1\) for every symplectic matrix \(S \in Sp(2n)\).

Throughout the paper you may use standard results of linear algebra without proof, provided that you state the results which you use clearly.
1 (i) Define what it means for a $2n \times 2n$ matrix $S$ to be symplectic. (2 marks)

(ii) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ matrix in block form, where $A, B, C$ and $D$ denote $n \times n$ matrices.
Prove that $S$ is symplectic if and only if the three equations

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I,$$

hold. (5 marks)

(iii) Now let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a symplectic $2n \times 2n$ matrix and assume that $\det A \neq 0$.
(a) Show that $T := \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$ is symplectic.
(b) Calculate $TS$, simplifying your answer.
(c) Hence, or otherwise, show that

$$\det S = \det A \det(D - CA^{-1}B).$$

(8 marks)

(iv) Let $S \in Sp(2n)$ and let

$$p(\lambda) = \det(S - \lambda I)$$

be the characteristic polynomial of $S$.
(a) Show that $p(\frac{1}{\lambda}) = \frac{1}{\lambda^{2n}}p(\lambda)$ for $\lambda \neq 0$. (8 marks)

(b) Using (a) or otherwise, show that if $\lambda$ is an eigenvalue for $S$, then $\frac{1}{\lambda}$ is also.
(You may assume that every eigenvalue of $S$ is non-zero.) (2 marks)
Given any vector space $V$, define the dual space $V^*$ of $V$, including a brief description of the vector space operations. (2 marks)

Now let $W \subseteq V$ be a vector subspace of $V$. Define the annihilator $W^o$ of $W$ in $V^*$. (2 marks)

Define the restriction map $\rho: V^* \to W^*$ and prove that $\rho$ is surjective. (You may assume without proof that $\rho$ is linear.) (8 marks)

Using (c), or otherwise, find the dimension of $W^o$ in terms of the dimensions of $V$ and $W$. (You may use, without proof, the fact that the dimension of the dual space $V^*$ is equal to the dimension of $V$, for any vector space $V$.) (7 marks)

Now let $(V, \omega)$ be a symplectic vector space and let $W \subseteq V$ be a vector subspace. Prove that
\[
\dim W^\wedge = \dim V - \dim W.
\] (6 marks)
3 (i) (a) Consider a beach, with a straight coastline, as shown in Figure 1. Person M, standing on the beach, sees person S in the water, struggling and getting into difficulties. M can run at speed $v_1$ and swim at speed $v_2$.

M, being a mathematician, runs towards a point P so as to reach S in the shortest possible time. Prove that the angles which M’s path on land and in the water will make with the normal to the beach are related by

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$  

![Figure 1: For Question 3(i)](image)

(b) In one or two sentences, explain how the result of (a) is related to Snell’s Law for refraction. (4 marks)

(ii) Let $S \in Sp(2n)$ be a symplectic matrix and write it in block form as $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A, B, C$ and $D$ denote $n \times n$ matrices.

(a) Let $L$ denote the vector subspace of $\mathbb{R}^{2n}$ generated by the first $n$ columns of $S$. Prove that $L$ is a Lagrangian subspace of $\mathbb{R}^{2n}$ with respect to $\Omega$. (8 marks)

(b) Let $L'$ denote the vector subspace of $\mathbb{R}^{2n}$ generated by the last $n$ columns of $S$. Prove that $\mathbb{R}^{2n} = L \oplus L'$. (4 marks)
In this question, we consider $\mathbb{R}^4$ and $\mathbb{R}^{2n}$ with the standard symplectic form $\Omega$.

(a) In $\mathbb{R}^4$ let $K = \{(q, 0, 0, 0) \mid q \in \mathbb{R}\}$. Find $K^\wedge$. (2 marks)

(b) In $\mathbb{R}^4$ let $U = \{(0, y, 0, z) \mid y, z \in \mathbb{R}\}$. Show that $U$ is a symplectic subspace of $\mathbb{R}^4$. (2 marks)

(c) Show that if $S \in Sp(2n)$ then $\Omega(SZ_1, SZ_2) = \Omega(Z_1, Z_2)$ for all $Z_1, Z_2 \in \mathbb{R}^{2n}$. (2 marks)

(d) Let $W \subseteq \mathbb{R}^{2n}$ be a subspace and let $S \in Sp(2n)$ be such that $SZ \in W$ for all $Z \in W$. Show that $SZ' \in W^\wedge$ for all $Z' \in W^\wedge$. (4 marks)

(e) Let $S \in Sp(4)$ be a symplectic matrix which maps the subspace $K = \{(q, 0, 0, 0) \mid q \in \mathbb{R}\}$ defined in (a) to itself. Show that $S(U) = U$ where $U$ is defined in (b). (5 marks)

Show that of the 16 entries of $S = [s_{ij}]$, ten are zero.

Show that $s_{11}s_{33} = 1$ and find another equation which the remaining four non-zero entries must satisfy. (10 marks)

End of Question Paper