The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Optics and Symplectic Geometry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. *The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.*

You may use without proof the fact that $\det S = 1$ for every symplectic matrix $S \in Sp(2n)$.

Throughout the paper you may use standard results of linear algebra without proof, provided that you state the results which you use clearly.

1 (i) Define what it means for a $2n \times 2n$ matrix S to be symplectic. **(2 marks)**

(ii) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold. **(5 marks)**

(iii) Now let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a symplectic $2n \times 2n$ matrix and assume that $\det A \neq 0$.

(a) Show that $T := \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix}$ is symplectic.

(b) Calculate TS, simplifying your answer.

(c) Hence, or otherwise, show that

$$\det S = \det A \, \det(D - CA^{-1}B).$$

(8 marks)

(iv) Let $S \in Sp(2n)$ and let

$$p(\lambda) = \det(S - \lambda I)$$

be the characteristic polynomial of S.

(a) Show that $p(\frac{1}{\lambda}) = \frac{1}{\lambda^{2n}} p(\lambda)$ for $\lambda \neq 0$.

(8 marks)

(b) Using (a) or otherwise, show that if λ is an eigenvalue for S, then $\frac{1}{\lambda}$ is also.

(You may assume that every eigenvalue of S is non-zero.)

(2 marks)

2 (a) Given any vector space V, define the dual space V^* of V, including a brief description of the vector space operations. **(2 marks)**

(b) Now let $W \subseteq V$ be a vector subspace of V. Define the *annihilator* W° of W in V^*. **(2 marks)**

(c) Define the *restriction map* $\rho\colon V^* \to W^*$ and prove that ρ is surjective.

(You may assume without proof that ρ is linear.) **(8 marks)**

(d) Using (c), or otherwise, find the dimension of W° in terms of the dimensions of V and W.

(You may use, without proof, the fact that the dimension of the dual space V^* is equal to the dimension of V, for any vector space V.) **(7 marks)**

(e) Now let (V, ω) be a symplectic vector space and let $W \subseteq V$ be a vector subspace. Prove that

$$\dim W^\wedge = \dim V - \dim W.$$

(6 marks)

3 (i) (a) Consider a beach, with a straight coastline, as shown in Figure 1. Person M, standing on the beach, sees person S in the water, struggling and getting into difficulties. M can run at speed v_1 and swim at speed v_2.

M, being a mathematician, runs towards a point P so as to reach S in the shortest possible time. Prove that the angles which M's path on land and in the water will make with the normal to the beach are related by

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

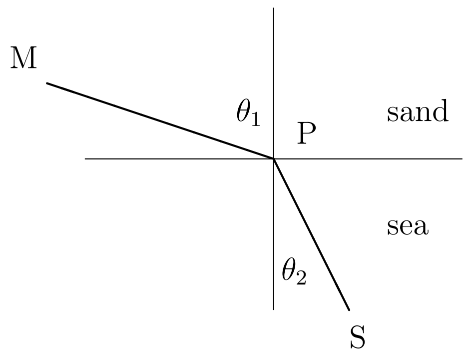

Figure 1: For Question 3(i)

(9 marks)

(b) In one or two sentences, explain how the result of (a) is related to Snell's Law for refraction. **(4 marks)**

(ii) Let $S \in Sp(2n)$ be a symplectic matrix and write it in block form as $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ where A, B, C and D denote $n \times n$ matrices.

(a) Let L denote the vector subspace of $\mathbb{R}^{2n}$ generated by the first n columns of S. Prove that L is a Lagrangian subspace of $\mathbb{R}^{2n}$ with respect to Ω. **(8 marks)**

(b) Let L' denote the vector subspace of $\mathbb{R}^{2n}$ generated by the last n columns of S. Prove that $\mathbb{R}^{2n} = L \oplus L'$. **(4 marks)**

4 In this question, we consider $\mathbb{R}^4$ and $\mathbb{R}^{2n}$ with the standard symplectic form Ω.

(a) In $\mathbb{R}^4$ let $K = \{(q, 0, 0, 0) \mid q \in \mathbb{R}\}$. Find $K^\wedge$. **(2 marks)**

(b) In $\mathbb{R}^4$ let $U = \{(0, y, 0, z) \mid y, z \in \mathbb{R}\}$. Show that U is a symplectic subspace of $\mathbb{R}^4$. **(2 marks)**

(c) Show that if $S \in Sp(2n)$ then $\Omega(SZ_1, SZ_2) = \Omega(Z_1, Z_2)$ for all $Z_1, Z_2 \in \mathbb{R}^{2n}$. **(2 marks)**

(d) Let $W \subseteq \mathbb{R}^{2n}$ be a subspace and let $S \in Sp(2n)$ be such that $SZ \in W$ for all $Z \in W$.

Show that $SZ' \in W^\wedge$ for all $Z' \in W^\wedge$. **(4 marks)**

(e) Let $S \in Sp(4)$ be a symplectic matrix which maps the subspace $K = \{(q, 0, 0, 0) \mid q \in \mathbb{R}\}$ defined in (a) to itself.

Show that $S(U) = U$ where U is defined in (b). **(5 marks)**

Show that of the 16 entries of $S = [s_{ij}]$, ten are zero.
Show that $s_{11}s_{33} = 1$ and find another equation which the remaining four non-zero entries must satisfy. **(10 marks)**

End of Question Paper