The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Galois Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper K denotes a subfield of $\mathbb{C}$ which contains $\mathbb{Q}$. All field extensions are finite.

1 Consider a reduced quartic polynomial $f(X) = X^4 + pX^2 + qX + r \in \mathbb{Q}[x]$.
Write α_1, α_2, α_3, α_4, for the roots of $f(X)$ in a field L which extends $\mathbb{Q}$.

(a) Prove that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. **(2 marks)**

(b) Write $\beta = \alpha_1 + \alpha_2$, $\gamma = \alpha_1 + \alpha_3$, $\delta = \alpha_1 + \alpha_4$.
Prove *any two* of the following formulas:

$$\alpha_1 = \tfrac{1}{2}(\beta+\gamma+\delta), \quad \alpha_2 = \tfrac{1}{2}(\beta-\gamma-\delta), \quad \alpha_3 = \tfrac{1}{2}(-\beta+\gamma-\delta), \quad \alpha_4 = \tfrac{1}{2}(-\beta-\gamma+\delta).$$

(4 marks)

(c) Prove *any one* of the following formulas:

$$\beta^2 + \gamma^2 + \delta^2 = -2p, \quad \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2 = p^2 - 4r, \quad \beta\gamma\delta = -q.$$

(8 marks)

(d) Show that β^2, γ^2 and δ^2 are the solutions of the cubic equation

$$Y^3 + 2pY^2 + (p^2 - 4r)Y - q^2 = 0.$$

(3 marks)

(e) In three or four sentences, explain how, given the solutions of the cubic equation in (d), you would find the solutions of the reduced quartic equation $f(X) = 0$. **(4 marks)**

(f) Solve the reduced quartic equation

$$X^4 + 2X^2 + 4X + 2 = 0.$$

(4 marks)

2 Write $\alpha = \sqrt{4+\sqrt{7}}$ and $\beta = \sqrt{4-\sqrt{7}}$. Write $L = \mathbb{Q}(\alpha)$.

(a) Determine the minimal polynomial $f(x)$ of α over $\mathbb{Q}$ and show that it has four roots, $\pm\alpha$ and $\pm\beta$. **(4 marks)**

(b) Show that $f(x)$ is irreducible, by using a shifted Eisenstein criterion, or otherwise. **(3 marks)**

(c) Show that $\beta \in L$ and deduce that L is a Galois extension of $\mathbb{Q}$ of degree 4. **(3 marks)**

(d) Write down all the elements of $\mathrm{Gal}(L/\mathbb{Q})$ and their effect on α and on β. Describe the group $\mathrm{Gal}(L/\mathbb{Q})$ in terms of standard groups. **(4 marks)**

(e) Find the images of α^2, $\alpha+\beta$ and $\alpha-\beta$ under each element of $\mathrm{Gal}(L/\mathbb{Q})$. Using the Fundamental Theorem of Galois Theory, pair $\mathbb{Q}(\alpha^2)$, $\mathbb{Q}(\alpha+\beta)$ and $\mathbb{Q}(\alpha-\beta)$ with appropriate subgroups of $\mathrm{Gal}(L/\mathbb{Q})$. **(6 marks)**

(f) Write down the subfield lattice of of all intermediate fields $\mathbb{Q} \subseteq M \subseteq L$. **(5 marks)**

3 Let $f(x) = x^4 + 8x^2 - 2 \in \mathbb{Q}[x]$, and define $\alpha = \sqrt{3\sqrt{2}-4}$. Write $L = \mathbb{Q}(\alpha, i\sqrt{2})$. You may assume that L is a splitting field for $f(x)$ and that $[L : \mathbb{Q}] = 8$.

(a) Show that the roots of f are $\pm\alpha$ and $\pm\frac{i\sqrt{2}}{\alpha}$. **(2 marks)**

(b) Compute the elements of $\mathrm{Gal}(L/\mathbb{Q})$, listing in a table the effect of each element on α and $i\sqrt{2}$. **(7 marks)**

(c) Show that $\mathrm{Gal}(L/\mathbb{Q})$ contains an automorphism φ of order 4 and an automorphism ψ of order 2, such that $\mathrm{Gal}(L/\mathbb{Q}) = \langle\varphi, \psi\rangle$. **(10 marks)**

(d) Write $\psi\varphi\psi^{-1}$ in the form $\varphi^i\psi^j$. To what well-known group is $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic? **(6 marks)**

4 (i) (a) Define what it means for a group G to be *soluble*. **(2 marks)**

(b) Prove that S_4 is soluble, being sure to verify all conditions. **(5 marks)**

(ii) Consider the group A_5. In this part you may use without proof the fact that elements x, x' of S_5 have the same cycle type if and only if they are conjugate in S_5.

(a) List the cycle types which non-identity elements of A_5 may have. **(3 marks)**

(b) Let $g, g' \in A_5$ have the same cycle type. Show that if there is an element $x \in S_5$ which has odd parity and commutes with g, then g and g' are conjugate in A_5. **(2 marks)**

(c) Show that all transposition pairs are conjugate in A_5 and all 3-cycles are conjugate in A_5. **(2 marks)**

Now suppose that N is a normal subgroup of A_5, not the identity subgroup.

(d) Suppose that N contains a 3-cycle. Show that N contains all transposition pairs. **(3 marks)**

(e) Suppose that N contains a transposition pair. Show that N contains all 3-cycles. **(2 marks)**

(f) Suppose that N contains a 5-cycle. Show that N contains all 3-cycles. **(3 marks)**

(g) Show that $N = A_5$. **(3 marks)**

End of Question Paper