The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2016–2017**

**MAS451 Measure and Probability** **2 hours 30 minutes**

*Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.*

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**1** (i) Given two *probability* measures $P_1$ and $P_2$ on a measurable space $(S, \Sigma)$ and a number $0 \leq c \leq 1$, define $P(A) = cP_1(A) + (1-c)P_2(A)$ for all $A \in \Sigma$. Show that $P$ is also a probability measure on $(S, \Sigma)$. ***(4 marks)***

(ii) (a) Let $(S, \Sigma, m)$ be a measure space with $m$ a finite measure, that is $m(S) < \infty$. Let $A_n, n \geq 1$ be an increasing sequence of subsets of $S$, that is $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Define $A := \cup_{n=1}^{\infty} A_n$. Show that

$$m(A) = \lim_{n\to\infty} m(A_n).$$

(HINT: Write $\cup_{n=1}^{\infty} A_n$ as a disjoint union.) ***(4 marks)***

(b) Let $(S, \Sigma, m)$ be a measure space with $m$ a finite measure, that is $m(S) < \infty$. Let $B_n, n \geq 1$ be a decreasing sequence of subsets of $S$, that is $B_{n+1} \subseteq B_n$ for all $n \geq 1$. Define $B := \cap_{n=1}^{\infty} B_n$. Show that

$$m(B) = \lim_{n\to\infty} m(B_n).$$

(HINT: Consider $A_n = S - B_n$ and use part (a).) ***(5 marks)***

(c) Give a counterexample to show that the above identity does not hold in general when the sets $B_n$ are decreasing and $m$ is an *infinite* measure. ***(2 marks)***

(iii) Consider the set $S = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Write down the smallest $\sigma$-algebra $\Sigma$ of $S$ which contains $A$ and $B$. ***(5 marks)***

(iv) Recall that the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma$-algebra of $\mathbb{R}$ that contains open intervals $(a, b)$, $-\infty \leq a < b \leq \infty$. Show that $\mathcal{B}(\mathbb{R})$ contains sets of the form $[a, b]$ and $\{a\}$, where $-\infty < a < b < \infty$. ***(5 marks)***

(v) For any set $A \subseteq \mathbb{R}$ define $-A := \{-x : x \in A\}$. Consider the collection

$$\mathcal{C} = \{A \in \mathcal{B}(\mathbb{R}) : -A \in \mathcal{B}(\mathbb{R})\}.$$

(a) Show that $\mathcal{C}$ is a $\sigma$-algebra. ***(5 marks)***

(b) Show that $\mathcal{C} = \mathcal{B}(\mathbb{R})$. ***(3 marks)***

**2** (i) Consider the sequence $a_n$, $n \geq 1$ given by

$$a_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ is odd} \\ -1 + \frac{1}{n}, & \text{if } n \text{ is even} \end{cases}$$

(a) Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. ***(3 marks)***

(b) Does $\lim_{n\to\infty} a_n a_{n+1}$ exist? If so, what is the limit? ***(2 marks)***

(ii) Let $(S, \Sigma)$ be a measurable space and let $f, g : S \to \mathbb{R}$ be two measurable functions.

(a) Show that $f + \mathbf{1}$ is a measurable function, where $\mathbf{1} : S \to \mathbb{R}$ is the function which is identically one, $\mathbf{1}(s) = 1$ for all $s \in S$. ***(3 marks)***

(b) Show that $\{f > g\} \in \Sigma$.
(HINT: If $f(x) > g(x)$ then there must be a rational point between $f(x)$ and $g(x)$) ***(4 marks)***

(iii) Give an example of a measurable space $(S, \Sigma)$ and a function $f : S \to \mathbb{R}$ such that $|f|$ is measurable but $f$ is not. ***(4 marks)***

(iv) Let $S$ be a set with $A \subseteq S$ and consider the $\sigma$-algebra $\Sigma = \{\emptyset, A, A^c, S\}$. Show that a function $f : S \to \mathbb{R}$ is measurable if and only if it is constant on $A$ and also constant on $A^c$.
(HINT: For one direction consider $f^{-1}(\{a\})$ for $a \in \mathbb{R}$.) ***(5 marks)***

(v) Let $(S, \Sigma, m)$ be a measure space and $f : S \to \mathbb{R}$ be an integrable function.

(a) Suppose $m(f \neq 0) = 0$. Show that $\int_S f\,dm = 0$.
(HINT: Write $f = f \cdot \mathbf{1}_{\{f=0\}} + f \cdot \mathbf{1}_{\{f\neq 0\}}$) ***(4 marks)***

(b) Compute $\int_0^1 \mathbf{1}_{\mathbb{Q}}(x)\,dx$. ***(1 mark)***

(c) Suppose $\int_S f\,dm = 0$. Is it true that $m(f \neq 0) = 0$? Prove it if it is true or else provide a counterexample if it is false. ***(3 marks)***

(vi) Consider $([0,2], \mathcal{B}([0,2]), \lambda)$ where $\lambda$ is the Lebesgue measure on $[0,2]$. Let $f : [0,2] \to \mathbb{R}$ be a nonnegative integrable function such that $f(x) \leq 3$ for $x \leq 1$ and $\int_1^2 f(x)\,dx = 2$. Show that ***(4 marks)***

$$\lambda\left(x : f(x) \geq 4\right) \leq \frac{1}{2}.$$

**3** (i) State the Dominated Convergence theorem and use it to find the limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} \left[1 - e^{-|x|/n}\right] \cdot e^{-|x|}\, dx.$$

***(5 marks)***

(ii) Let $(S, \Sigma, m)$ be a measure space. The *Monotone Convergence theorem* states that for any monotonic increasing sequence of non negative measurable functions $f_n$ from $S$ to $\mathbb{R}$ we have

$$\int_S \lim_{n\to\infty} f_n\, dm = \lim_{n\to\infty} \int_S f_n\, dm.$$

Give a counterexample to show that the above identity does not hold if the functions $f_n$ are monotonic decreasing. ***(3 marks)***

(iii) Let $(S, \Sigma, m)$ be a measure space. Let $(f_n)$ be a sequence of non-negative measurable functions for which $f_n \le f$ for all $n \in \mathbb{N}$ where $f$ is integrable. Prove that

$$\limsup_{n\to\infty} \int_S f_n\, dm \le \int_S \limsup_{n\to\infty} f_n\, dm.$$

(HINT: Apply Fatou's lemma to $f - f_n$.) ***(5 marks)***

(iv) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X$ be a random variable that takes positive *intege*r values.

(a) Deduce that $X = \sum_{i=1}^{\infty} \mathbf{1}_{\{X \ge i\}}$.
(HINT: Consider the event $\{X(\omega) = k\}$) ***(3 marks)***

(b) Show that $\mathbb{E}(X) = \sum_{i=1}^{\infty} P(X \ge i)$. ***(3 marks)***

(v) Prove that in any infinite sequence of independent (fair) coin tosses, the pattern $HTHHT$ appears infinitely often, where $H$ represents heads and $T$ represents tails. ***(4 marks)***

(vi) Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\lambda$ is the uniform measure on $[0,1]$. Show that the random variables $X_n = n \cdot \mathbf{1}_{(0,n^{-1})}$ converge almost surely to $X \equiv 0$ but $X_n$ does not converge in mean square to $X$. ***(4 marks)***

(vii) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean 0 and variance 1. Let $S_n = X_1 + X_2 + \cdots + X_n$, $n \ge 1$ and consider the event $A_n = \{S_n \in [1,2]\}$. Show that

$$\mathbb{E}\left[\left(S_{n+1}^2 - (n+1)\right) \cdot \mathbf{1}_{A_n}\right] = \mathbb{E}\left[\left(S_n^2 - n\right) \cdot \mathbf{1}_{A_n}\right]$$

***(6 marks)***

**4** (i) Let $(S_1, \Sigma_1, m_1)$ and $(S_2, \Sigma_2, m_2)$ be measure spaces. Recall that for $E \subseteq S_1 \times S_2$ and $x \in S_1$ the $x$-slice of $E$ is

$$E_x := \{y \in S_2 : (x, y) \in E\}.$$

Let $E, F \subseteq S_1 \times S_2$ and $x \in S_1$. Show that

(a) $(E \cap F)_x = E_x \cap F_x$. ***(3 marks)***

(b) $(E^c)_x = (E_x)^c$ ***(3 marks)***

(c) $(\cup_{n=1}^{\infty} E_n)_x = \cup_{n=1}^{\infty} (E_n)_x$ where $E_n$, $n \geq 1$ is a sequence of subsets of $S_1 \times S_2$. ***(3 marks)***

(ii) State the version of Fubini's theorem for nonnegative measurable functions. ***(4 marks)***

(iii) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X$ be a nonnegative random variable with $0 \leq X \leq 1$. Consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$. Consider the product space $\Omega \times [0,1]$ with product $\sigma$-algebra and product probability $P \times \lambda$.

(a) Show that the set $G$ is in the product $\sigma$-algebra, where

$$G = \{(\omega, y) : y \leq X(\omega)\}.$$

(HINT: Consider $G^c$ and note that if $X(\omega) < y$ then there must be a rational number between $X(\omega)$ and $y$) ***(5 marks)***

(b) Show that $P \times \lambda(G) = E(X)$. ***(5 marks)***

(iv) (a) Let $(S, \Sigma)$ be a measurable space and let $m_1$ and $m_2$ be two finite measures on it with the property $m_1(S) = m_2(S)$. Show that the collection

$$\mathcal{C} := \{A \in \Sigma : m_1(A) = m_2(A)\}$$

is a $\lambda$-system. ***(5 marks)***

(b) Show that the Lebesgue measure is the only measure $m$ on the Borel sets of the interval $[0,1]$ with the property that for all subintervals $J$, $m(J) =$ length of $J$.
(HINT: Use Dynkin's $\pi - \lambda$ theorem) ***(5 marks)***

## End of Question Paper