The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

Financial Mathematics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Consider the following four bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
0.5	0	99.50
1	6%	104.71
1.5	10%	112.58
1.5	8%	?

(a) Use the bootstrap method to find the 0.5, 1 and 1.5-year spot interest rates. ***(9 marks)***

(b) Find the price of the fourth bond in the table above. ***(4 marks)***

(ii) Consider a twelve-month forward contract on 1,000,000 shares of *Too Big To Fail Plc.* These shares are currently traded for £80 per share. Within the next twelve months, *Too Big To Fail Plc* will pay a single dividend of £3 per share in 6 months. The six-month and twelve-month spot interest rates are 2% and 3%, respectively.

(a) What is the correct forward price for this forward contract? ***(2 marks)***

(b) You are given the opportunity to take a long or a short position in this forward contract at a forward price of £79,000,000. Describe in detail an arbitrage opportunity available to you. ***(10 marks)***

2 (i) Let $0 \le X_1 < X_2$ and let $0 \le \lambda \le 1$. Consider three European call options on the same underlying asset and with the same expiration time T. The strike prices of these are X_1, X_2 and $\lambda X_1 + (1-\lambda)X_2$ and their corresponding spot prices are denoted c_1, c_2 and c.

For any $0 \le \lambda \le 1$ consider portfolio Π_λ which is

- long λ call options with strike X_1,
- long $1-\lambda$ call options with strike X_2,
- short one call option with strike $\lambda X_1 + (1-\lambda)X_2$.

(a) Sketch the graph of the payoff of $\Pi_{\frac{1}{2}}$ as a function of the underlying asset price S_T at expiration. ***(4 marks)***

(b) Show that for any S,

$$\max\{S-\lambda X_1-(1-\lambda)X_2, 0\} \le \lambda \max\{S-X_1, 0\}+(1-\lambda)\max\{S-X_2, 0\}.$$

You may want to show that this holds in the following three cases separately: $S < X_1$, $X_1 \le S \le X_2$, $X_2 < S$ and use the fact that $X_1 \le \lambda X_1 + (1-\lambda)X_2 \le X_2$. ***(6 marks)***

(c) Write down the payoff function of Π_λ at expiration and use part (b) to deduce an inequality satisfied by the payoff. ***(3 marks)***

(d) Use part (c) to deduce an inequality between c_1, c_2 and c. ***(6 marks)***

(ii) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 2%.

Use a binomial tree to find the price of a two-year American put option on this stock with strike price £6. ***(6 marks)***

3 (i) (a) State Ito's Lemma. ***(3 marks)***

(b) Let $\{B_t\}_{t\geq 0}$ denote Brownian motion. Describe the stochastic process $\{B_t^2\}_{t\geq 0}$. ***(4 marks)***

(ii) Consider two assets whose prices S_1 and S_2 follow the Ito processes

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dB, \quad dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dB$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 \geq 0$ and B denotes Brownian motion. Assume that all risk-free interest rates are constant and equal to r.

(a) Describe a portfolio Π involving these assets whose value after an infinitesimal period of time is known in advance. Explain your answer. ***(6 marks)***

(b) At time t you invest in one unit of Π; what is the approximate value of your investment after a very short period of time Δt. ***(2 marks)***

(c) At time t you invest the value of Π in a risk-free investment; what is the approximate value of your investment after a very short period of time Δt. ***(4 marks)***

(d) Use (b) and (c) to describe an identity involving $\mu_1, \mu_2, \sigma_1, \sigma_2$ and r. ***(6 marks)***

4 (i) Explain the following terms in the context of portfolio theory:

(a) *feasible set*, ***(2 marks)***

(b) *efficient frontier*, and ***(2 marks)***

(c) *market portfolio.* ***(2 marks)***

(ii) Consider a market with only two risky investments A and B. Let their expected returns be 12% and 20%, respectively. Their standard deviation of returns are 10% and 20%, respectively. The correlation between the returns of A and B is 0.75.

(a) Assuming there is no risk-free investment, find the feasible set as a curve in the σ-r plane given in parametric form. What is the efficient frontier? ***(7 marks)***

(b) Assume that a risk-free investment in this world exists and has risk-free return equal to 3%. Find the market portfolio. ***(12 marks)***

End of Question Paper