Financial Mathematics

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student
1 (i) Consider the following four bonds with face value of £100:

<table>
<thead>
<tr>
<th>Time to maturity (in years)</th>
<th>Annual interest (paid every 6 months)</th>
<th>Bond price (in £)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0</td>
<td>99.50</td>
</tr>
<tr>
<td>1</td>
<td>6%</td>
<td>104.71</td>
</tr>
<tr>
<td>1.5</td>
<td>10%</td>
<td>112.58</td>
</tr>
<tr>
<td>1.5</td>
<td>8%</td>
<td>?</td>
</tr>
</tbody>
</table>

(a) Use the bootstrap method to find the 0.5, 1 and 1.5-year spot interest rates.  
(9 marks)

(b) Find the price of the fourth bond in the table above.  
(4 marks)

(ii) Consider a twelve-month forward contract on 1,000,000 shares of Too Big To Fail Plc. These shares are currently traded for £80 per share. Within the next twelve months, Too Big To Fail Plc will pay a single dividend of £3 per share in 6 months. The six-month and twelve-month spot interest rates are 2% and 3%, respectively.

(a) What is the correct forward price for this forward contract?  
(2 marks)

(b) You are given the opportunity to take a long or a short position in this forward contract at a forward price of £79,000,000. Describe in detail an arbitrage opportunity available to you.  
(10 marks)
Let $0 \leq X_1 < X_2$ and let $0 \leq \lambda \leq 1$. Consider three European call options on the same underlying asset and with the same expiration time $T$. The strike prices of these are $X_1$, $X_2$ and $\lambda X_1 + (1-\lambda)X_2$ and their corresponding spot prices are denoted $c_1$, $c_2$ and $c$.

For any $0 \leq \lambda \leq 1$ consider portfolio $\Pi_\lambda$ which is

- long $\lambda$ call options with strike $X_1$,
- long $1-\lambda$ call options with strike $X_2$,
- short one call option with strike $\lambda X_1 + (1-\lambda)X_2$.

(a) Sketch the graph of the payoff of $\Pi_\lambda$ as a function of the underlying asset price $S_T$ at expiration. (4 marks)

(b) Show that for any $S$,

\[
\max\{S-\lambda X_1-(1-\lambda)X_2,0\} \leq \lambda \max\{S-X_1,0\}+(1-\lambda) \max\{S-X_2,0\}.
\]

You may want to show that this holds in the following three cases separately: $S < X_1$, $X_1 \leq S \leq X_2$, $X_2 < S$ and use the fact that $X_1 \leq \lambda X_1 + (1-\lambda)X_2 \leq X_2$. (6 marks)

(c) Write down the payoff function of $\Pi_\lambda$ at expiration and use part (b)

to deduce an inequality satisfied by the payoff. (3 marks)

(d) Use part (c) to deduce an inequality between $c_1$, $c_2$ and $c$. (6 marks)

(ii) The price of a stock which pays no dividends is currently £8. Over each of the next two 1-year periods the stock price will either increase by 50% or decrease by 50%. Suppose that all interest rates are constant and equal to 2%.

Use a binomial tree to find the price of a two-year American put option on this stock with strike price £6. (6 marks)
3 (i) (a) State Ito’s Lemma. (3 marks)
(b) Let \( \{B_t\}_{t \geq 0} \) denote Brownian motion. Describe the stochastic process \( \{B_t^2\}_{t \geq 0} \). (4 marks)

(ii) Consider two assets whose prices \( S_1 \) and \( S_2 \) follow the Ito processes
\[
dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dB, \quad dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dB
\]
where \( \mu_1, \mu_2, \sigma_1, \sigma_2 \geq 0 \) and \( B \) denotes Brownian motion. Assume that all risk-free interest rates are constant and equal to \( r \).
(a) Describe a portfolio \( \Pi \) involving these assets whose value after an infinitesimal period of time is known in advance. Explain your answer. (6 marks)
(b) At time \( t \) you invest in one unit of \( \Pi \); what is the approximate value of your investment after a very short period of time \( \Delta t \). (2 marks)
(c) At time \( t \) you invest the value of \( \Pi \) in a risk-free investment; what is the approximate value of your investment after a very short period of time \( \Delta t \). (4 marks)
(d) Use (b) and (c) to describe an identity involving \( \mu_1, \mu_2, \sigma_1, \sigma_2 \) and \( r \). (6 marks)

4 (i) Explain the following terms in the context of portfolio theory:
(a) feasible set, (2 marks)
(b) efficient frontier, and (2 marks)
(c) market portfolio. (2 marks)

(ii) Consider a market with only two risky investments \( A \) and \( B \). Let their expected returns be 12% and 20%, respectively. Their standard deviation of returns are 10% and 20%, respectively. The correlation between the returns of \( A \) and \( B \) is 0.75.
(a) Assuming there is no risk-free investment, find the feasible set as a curve in the \( \sigma-r \) plane given in parametric form. What is the efficient frontier? (7 marks)
(b) Assume that a risk-free investment in this world exists and has risk-free return equal to 3%. Find the market portfolio. (12 marks)

End of Question Paper