The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS6340: Analysis I **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$.

1 (i) Say what is meant by the statement that a normed vector space is a Banach space. ***(2 marks)***

(ii) Let X be a compact metric space. Prove that the space $C(X)$ of all continuous functions from the space X to the field $\mathbb{C}$ is a Banach space. ***(9 marks)***

(iii) Say what is meant by a closed subset of a Banach space. ***(2 marks)***

(iv) Let X be a compact metric space. State the Stone-Weierstrass theorem on subalgebras of $C(X)$. ***(3 marks)***

(v) Which of the following subspaces of $C([0,1])$ are closed? Justify your answer.

(a) The set of all polynomials of degree 3.

(b) The set of all functions $f \in C([0,1])$ such that $f(0) = 0$.

(c) The set of all polynomials.

(9 marks)

2 (i) Let V be a complex normed vector space. Say what is meant by the statement that a linear map $f: V \to \mathbb{C}$ is a bounded linear map, and define the norm of such a map. ***(2 marks)***

(ii) Which of the following are bounded linear maps? For those which are bounded linear maps, calculate their norms.

(a)

$$I: C([0,1]) \to \mathbb{C} \qquad I(f) = \int_0^1 f(t)\ dt;$$

(b)

$$E: C([0,1]) \to \mathbb{C} \qquad E(f) = 2f(0);$$

(c)

$$D: C^\infty([0,1]) \to \mathbb{C} \qquad D(f) = f'(1).$$

(10 marks)

(iii) State the Hahn-Banach theorem. ***(2 marks)***

(iv) Let V^* be the normed vector space of linear functionals on a normed vector space V. Prove that V^* is complete with respect to the norm you defined in part (i). ***(6 marks)***

(v) Define a linear map $\tau: V \to (V^*)^*$ by the formula

$$\tau(v)(f) = f(v) \qquad f \in V^*,\ v \in V.$$

Use the Hahn-Banach theorem to prove that $\|\tau(v)\| = \|v\|$ for all $v \in V$. ***(5 marks)***

3 (i) Let H be a Hilbert space. Let $T: H \to H$ be a bounded linear map. Define the adjoint, T^* of T, and prove that it is unique. ***(4 marks)***

(ii) Let $k: \mathbb{R}^2 \to \mathbb{C}$ be a compactly supported continuous function. Show that we have a bounded linear map $T_k: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by the formula

$$T_k(f)(s) = \int_{-\infty}^{\infty} k(x,t) f(t)\ dt.$$

Find the adjoint T_k^*. ***(8 marks)***

(iii) State the closed graph theorem on the continuity of linear maps between Banach spaces. ***(3 marks)***

(iv) Let $T: H \to H$ be a linear map. Suppose we have a linear map $S: H \to H$ such that

$$\langle S(u), v\rangle = \langle u, T(v)\rangle$$

for all $u, v \in H$. Prove that T is a bounded linear map. ***(10 marks)***

4 (i) Let A be a unital complex Banach algebra. Define the spectrum, $Spectrum(x)$, of an element $x \in A$, and prove that if $\lambda \in Spectrum(x)$ then $|\lambda| \leq \|x\|$. You may use without proof the fact that if $\|y\| < 1$ then the element $1 - y$ is invertible. ***(7 marks)***

(ii) State the spectral mapping theorem for polynomials. ***(3 marks)***

(iii) Let H be a Hilbert space, and let V be a closed subspace where $V \neq \{0\}$ and $V \neq H$. Let $P\colon H \to H$ be the orthogonal projection onto V.

(a) Show that $P^2 = P$. ***(3 marks)***

(b) Show that $Spectrum(P) = \{0, 1\}$. ***(8 marks)***

(c) Find $Spectrum(I + 3P^4)$. ***(4 marks)***

5 (i) State the definition of a linear map $K\colon V \to V$ on a normed vector space V being a compact operator. ***(2 marks)***

(ii) Prove that any bounded linear map $K\colon V \to V$ with finite-dimensional image is a compact operator. You may use the Heine-Borel theorem without proof. ***(4 marks)***

(iii) Which of the following linear maps on the space l^2 are compact operators? Justify your answer. You may use without proof the theorem that a norm-limit of compact operators is a compact operator.

(a) The identity map $I\colon l^2 \to l^2$.

(b) The right shift map $R\colon l^2 \to l^2$ defined by the formula

$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$

(c) The map $S\colon l^2 \to l^2$ defined by the formula

$$S(a_1, a_2, a_3, \ldots) = \left(a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots\right).$$

(10 marks)

(iv) Let H be a Hilbert space. Define what is meant by a Fredholm operator $T\colon H \to H$ and its index $Index(T)$. ***(4 marks)***

(v) Prove that the operator $R + S$, where R and S are the operators in part (iii) is Fredholm and find its index. You may use without proof any standard results about Fredholm operators from the lectures but you should state these in your proof. ***(6 marks)***

End of Question Paper