The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Spring Semester 2016-2017

Analytical Dynamics and Classical Field Theory **3 Hours**

Answer five questions. If you answer more than five questions, only your best five will be counted.

1 (i) Write down the Euler–Lagrange equation for the Lagrangian $L = L(q, \dot{q}, t)$, describing the dynamics of a particle moving on one dimension with position q and velocity $\dot{q} = dq/dt$. Define all quantities appearing in the equation. ***(4 marks)***

(ii) Let L be a Lagrangian which obeys the Euler–Lagrange equation. Consider the modified Lagrangian $\tilde{L} = L + df/dt$, where $f = f(q, t)$ is a function of the coordinate q and time t. Show that the Lagrangian $\tilde{L}$ also obeys the Euler–Lagrange equation by explicitly calculating the relevant derivatives. ***(8 marks)***

(iii) Define the action functional and state Hamilton's principle. ***(4 marks)***

(iv) Use Hamilton's principle to show that $\tilde{L}$ defined above obeys the Euler–Lagrange equation. ***(4 marks)***

2 In this question, a system of N particles is described by a Lagrange–function $L(q_i, \dot{q}_i, t)$, with $i = 1, \ldots, N$.

(i) Define the canonical momenta, the Hamilton function and state Hamilton's equations. ***(5 marks)***

(ii) Consider two functions f and g, which are functions of q_i, the canonical momenta P_i and time t. Write down the definition of the Poisson bracket $[f, g]$ between f and g. Show that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H] \,.$$

(5 marks)

(iii) In the following A, B and C are functions of q_i and P_i. Show that

(a) $[q_i, P_j] = \delta_{ij}$,

(b) $[A, q_i] = -\dfrac{\partial A}{\partial P_i}$,

(c) $[A, BC] = [A, B]C + [A, C]B$,

(10 marks)

3 In this question, you deal with Minkowski spacetime. You are given that the Lorentz–transformation between two internal systems with co-ordinates x^μ (System A) and x'^μ (System B) is given by $\Lambda^\mu{}_\nu = \partial x'^\mu / \partial x^\nu$. You are given that the metric tensor in Minkowski space is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and that the proper time is defined by $c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

(i) How do a four-vector x^μ and a tensor $\sigma^{\mu\nu}$ transform under Lorentz-transformations? ***(5 marks)***

(ii) The four-velocity is defined as $u^\mu = dx^\mu / d\tau$ and the four-acceleration is defined as $a^\mu = du^\mu / d\tau$, where $x^\mu = (ct, \vec{x})$. Show that the four-velocity is given by

$$u^\mu = \left(\frac{c}{\sqrt{1 - (v/c)^2}}, \frac{\vec{v}}{\sqrt{1 - (v/c)^2}} \right),$$

with $\vec{v} = d\vec{x}/dt$. Show that the four-acceleration can be written as

$$a^\mu = \left(\frac{(\vec{v}/c) \cdot (d\vec{v}/dt)}{(1 - (v/c)^2)^2}, \frac{d\vec{v}/dt}{1 - (v/c)^2} + \frac{(\vec{v}/c)((\vec{v}/c) \cdot (d\vec{v}/dt))}{(1 - (v/c)^2)^2} \right)$$

(10 marks)

(iii) Determine $u_\mu u^\mu$ and show that $u_\mu a^\mu = 0$. ***(5 marks)***

4 Let A_μ be the covariant components of a vector and $B_{\mu\nu}$ be the components of an antisymmetric tensor. The covariant derivative of a 2-tensor is given by $\mathcal{C}_{\mu\nu;\lambda} = \partial_\lambda \mathcal{C}_{\mu\nu} - \Gamma^\kappa_{\lambda\mu}\mathcal{C}_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu}\mathcal{C}_{\mu\kappa}$ and similarly for the covariant derivative of a four-vector.

(i) Show that

$$A_{\mu;\nu} - A_{\nu;\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

and

$$B_{\mu\nu;\rho} + B_{\nu\rho;\mu} + B_{\rho\mu;\nu} = \partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu}.$$

(7 marks)

(ii) Write down one symmetry property of the tensor $C_{\mu\nu\rho} = B_{\mu\nu;\rho} + B_{\nu\rho;\mu} + B_{\rho\mu;\nu}$ ***(1 mark)***

(iii) Write down the transformation rules for contravariant vectors A^μ and covariant vectors A_μ under general coordinate transformations $x^\mu \to \tilde{x}^\mu$. By demanding that the covariant derivative $A_{\mu;\nu}$ transforms as a tensor, show that the Christoffel symbols transform as

$$\tilde{\Gamma}^\lambda_{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \tilde{x}^\mu}\frac{\partial x^\gamma}{\partial \tilde{x}^\nu}\Gamma^\alpha_{\beta\gamma}(x) + \frac{\partial \tilde{x}^\lambda}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial \tilde{x}^\mu \tilde{x}^\nu}.$$

(12 marks)

5 You are given that the Riemann tensor obeys the following equation:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \ .$$

You are also given that $R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\kappa\nu}$. The Ricci tensor is defined by $R_{\mu\nu} = g^{\lambda\alpha} R_{\lambda\mu\alpha\nu}$ and the Ricci scalar by $R = g^{\mu\nu} R_{\mu\nu}$, where $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$.

(i) Use the equation above and the fact that the covariant derivative of the metric tensor vanishes (i.e. $g_{\mu\nu;\lambda} = 0$) to prove that

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\mu} = 0.$$

What does this equation imply for the energy momentum tensor $T^{\mu\nu}$? ***(13 marks)***

(ii) Einstein's equation for the gravitational field without the cosmological constant is given by

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \ ,$$

where $G_{\mu\nu}$ is the Einstein–tensor and $T_{\mu\nu}$ denotes the energy-momentum tensor. Show that Einstein's equation can be written as

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ ,$$

where $T = g^{\mu\nu} T_{\mu\nu} = T^{\mu}_{\ \mu}$ is the trace of the energy-momentum tensor. Show that in vacuum (i.e. in the case where the energy-momentum tensor vanishes), the Ricci tensor and the Ricci–scalar is zero. Show that the Ricci–scalar is also zero if the trace of the energy–momentum tensor vanishes. ***(7 marks)***

6 You are given the following: The metric tensor for an isotropic, homogeneous and spatially flat universe is given by:

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j,$$

where δ_{ij} is the Kronecker-symbol and a is a general function of time t. For this metric, the non-zero components of the Ricci-tensor are given by (the dot denotes a derivative with respect to t)

$$\begin{aligned} R_{00} &= 3\frac{\ddot{a}}{a}, \\ R_{ij} &= -(\ddot{a}a + 2\dot{a}^2)\delta_{ij} \ . \end{aligned}$$

You are also given that the non-zero components of the energy–momentum tensor are given by $T_{00} = \rho$ and $T_{ij} = a^2 p \, \delta_{ij}$, where ρ is the energy–density and p the pressure.

(i) Write down the components of the Einstein–tensor. ***(8 marks)***

(ii) Show that Einstein's field equation lead to

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho \ , \text{and} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \ . \end{aligned}$$

(8 marks)

(iii) Show that the equations in part (ii) imply that

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \ .$$

(4 marks)

End of Question Paper