



*Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.*

- 1 The Fourier transform,  $\hat{f}(k)$ , of a function  $f(x)$  is defined by

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{ikx} f(x) dx.$$

- (a) Using the above definition, derive the result

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) dk. \quad (8 \text{ marks})$$

$$\left[ \text{You may assume that } \int_{-\infty}^{\infty} e^{ikx} dk = 2\pi\delta(x). \right]$$

- (b) Show that the Fourier transform of

$$f(x) = x e^{-|x|}$$

is given by

$$\hat{f}(k) = \frac{4ik}{(1+k^2)^2}. \quad (10 \text{ marks})$$

By applying the inverse Fourier transform to  $\hat{f}(k)$ , deduce that for real  $x$

$$\int_0^{\infty} \frac{k \sin kx}{(1+k^2)^2} dk = \frac{\pi}{4} x e^{-|x|}. \quad (7 \text{ marks})$$

2 The function  $x(t)$  satisfies the ordinary differential equation

$$\ddot{x} + 2\dot{x} + 2x = f(t)$$

for  $t \geq 0$ , for some function  $f(t)$ , with  $x(0) = -1$  and  $\dot{x}(0) = 3$ .

(a) Taking the Laplace transform of the equation, find  $\tilde{x}(s)$  in terms of  $\tilde{f}(s)$ , where the Laplace transform  $\tilde{x}(s)$  is defined by

$$\tilde{x}(s) = \int_0^\infty e^{-st} x(t) dt,$$

and  $\tilde{f}(s)$  is defined similarly. (5 marks)

Hence derive the solution

$$x(t) = e^{-t}(2 \sin t - \cos t) + \int_0^t f(u) e^{-(t-u)} \sin(t-u) du. \quad (5 \text{ marks})$$

You may assume that the following hold:

$$\mathcal{L}\{x^{(n)}(t)\} = s^n \tilde{x}(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \dots - x^{(n-1)}(0)$$

$$\mathcal{L}\{e^{at}g(t)\} = \tilde{g}(s-a)$$

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} \quad \text{and} \quad \mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\left\{\int_0^t f(u)g(t-u) du\right\} = \tilde{f}(s)\tilde{g}(s),$$

where  $\mathcal{L}\{\cdot\}$  denotes the Laplace transform.

(b) Use the result of part (a) to find the solution  $x(t)$  when  $f(t) = e^{2t}$ . (8 marks)

Verify that this solution does satisfy the differential equation and the initial conditions. (7 marks)

- 3 The function  $y(x)$  satisfies the ordinary differential equation

$$y'' - y = \ln(1 + x) \quad 0 \leq x \leq 1, \quad (1)$$

with the boundary conditions

$$y = 0 \quad \text{at } x = 0 \quad \text{and at } x = 1.$$

- (a) Find the independent solutions of

$$y'' - y = 0. \quad (3 \text{ marks})$$

- (b) Given that Green's function  $G(x; \xi)$  for the boundary-value problem given at the beginning of the question is continuous at  $x = \xi$ , and that  $\partial G / \partial x$  has a discontinuity of size 1 at  $x = \xi$ , show that

$$G(x; \xi) = \begin{cases} \frac{\sinh(\xi - 1) \sinh x}{\sinh 1} & 0 \leq x < \xi, \\ \frac{\sinh \xi \sinh(x - 1)}{\sinh 1} & \xi < x \leq 1. \end{cases} \quad (14 \text{ marks})$$

- (c) Use Green's function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the  $\xi$  integrals). (3 marks)

Use this to find  $y'(x)$ , and hence to show that

$$y'(0) = \frac{1}{\sinh 1} \int_0^1 \sinh(\xi - 1) \ln(1 + \xi) d\xi. \quad (5 \text{ marks})$$

4 Consider the equation

$$\epsilon x^3 - 2x^2 + 18 = 0, \quad (2)$$

where  $\epsilon$  is a constant satisfying  $0 < \epsilon \ll 1$ .

(a) The solution can be written as

$$x = \frac{1}{\epsilon} (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots),$$

where  $x_0, x_1, x_2, \dots$  are  $O(1)$  as  $\epsilon \rightarrow 0$ .

Substitute into equation (2) to derive the solutions for  $x$ , correct to order  $\epsilon$  as  $\epsilon \rightarrow 0$ . **(19 marks)**

(b) Given the rearrangement

$$x = \frac{2}{\epsilon} \left( 1 - \frac{9}{x^2} \right)$$

of (2), use iteration to find the solution close to  $\frac{2}{\epsilon}$ , correct to order  $\epsilon^3$  as  $\epsilon \rightarrow 0$ . **(6 marks)**

5 The *exponential integral* is defined by

$$E(x) = \int_1^{\infty} t^{-1} e^{-xt} dt \quad \text{for } x > 0.$$

(a) Show, by changing variables, that

$$e^x E(x) = \int_0^{\infty} \frac{e^{-xv}}{1+v} dv. \quad (3 \text{ marks})$$

Use the sum of a geometric progression to show that

$$\frac{1}{1+v} = 1 - v + v^2 - v^3 + \dots + (-v)^{n-1} + \frac{(-v)^n}{1+v}. \quad (3 \text{ marks})$$

By using the above results and considering

$$I_n(x) = \int_0^{\infty} v^n e^{-xv} dv \quad \text{for } n = 0, 1, 2, \dots$$

deduce that

$$e^x E(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \dots + \frac{(-1)^{n-1}(n-1)!}{x^n} + R_n(x),$$

where

$$R_n(x) = (-1)^n \int_0^{\infty} \frac{v^n e^{-xv}}{1+v} dv. \quad (11 \text{ marks})$$

(b) By considering

$$\left| \frac{R_n(x)}{\frac{(-1)^{n-1}(n-1)!}{x^n}} \right|$$

as  $x \rightarrow \infty$ , show that  $E(x)$  has the asymptotic series

$$E(x) \sim e^{-x} \left( \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \dots + \frac{(-1)^n n!}{x^{n+1}} + \dots \right)$$

as  $x \rightarrow \infty$ .

(8 marks)

**End of Question Paper**