The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

Mathematics Core 1 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: ***55***

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

Section A:

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(22 marks)***

A1 If $A = \{1\}$ and $B = \{2\}$, then $A \times B$ is

A. $(1, 2)$ **B.** $\{1, 2\}$ **C.** $\{(1, 2)\}$ **D.** $\{(1, 2), (2, 1)\}$

A2 How many injective functions from $\{1, 2, 3\}$ to $\{4, 5, 6, 7, 8\}$ are there?

A. 125 **B.** 15 **C.** 3^5 **D.** 60

A3 Functions $f, g, h : \mathbb{R} \to \mathbb{R}$ are given by

$$f(x) = x^2 + 2x + 1, \quad g(x) = \sin(\pi x), \quad \text{and} \quad h = g \circ f.$$

What is $h(2017^{2017})$?

A. ∞ **B.** 2017 **C.** 0 **D.** 1

A4 How many different anagrams does `ABCABCABC` have?

A. $\frac{9!}{3^3}$ **B.** $\frac{9!}{3!}$ **C.** $\frac{9!}{3!3!3!}$ **D.** $\frac{9^9}{3^3 3^3 3^3}$

A5 If $r \neq 1$ then $r^6 + r^7 + r^8 + r^9 + r^{10}$ is equal to

A. $(r + r^2)^5$ **B.** $\frac{r^6(1 - r^5)}{1 - r}$ **C.** $\frac{r^5(1 + r^2)^5}{1 + r^5}$ **D.** $\frac{1 - r^{10}}{1 - r^2}$

A6 If $(1+2x+3x^2+4x^3)^{100} = a_0 + a_1 x + a_2 x^2 + \ldots$ then the series $a_0 + a_1 + a_2 + \ldots$ sums to

A. ∞ **B.** 0 **C.** 10^{100} **D.** 100

A7 $\lim\limits_{x\to 1} \dfrac{x^{100}-1}{x^2-1} =$

A. 100 **B.** 50 **C.** 1 **D.** ∞

A8 The function $f : \mathbb{R} \to \mathbb{R}$ has the property that $x \leq f(x) \leq x + 10$ for $x > 0$. Then $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} =$

A. 0 **B.** 1 **C.** 10 **D.** ∞

A9 If $e^y = \sin x$ for $0 < x < \pi$ then $\dfrac{dy}{dx} =$

A. $\tan x$ **B.** $\cot x$ **C.** $-\tan x$ **D.** $3 - \cot x$

A10 $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x}\, dx =$

A. $\ln \dfrac{\pi}{6}$ **B.** $\ln 2$ **C.** $\ln \sqrt{2}$ **D.** e

A11 $1 + \dfrac{1+3i}{1-2i} =$

A. $2-i$ **B.** $-i$ **C.** 0 **D.** i

A12 $e^{i\pi/6} =$

A. $\dfrac{\sqrt{3}+i}{2}$ **B.** $\dfrac{\sqrt{3}-i}{2}$ **C.** $\dfrac{-\sqrt{3}+i}{2}$ **D.** $\dfrac{\sqrt{3}-i}{2}$

A13 If $z = 3 + 4i$ then $|z^4| + |\overline{z}^4| =$

A. 25 **B.** 50 **C.** 1000 **D.** 1250

A14 If $\arg(z) = \dfrac{\pi}{3}$ and $\arg\left(\dfrac{1}{w}\right) = \dfrac{\pi}{4}$, then $\arg(zw) =$

A. $\dfrac{7\pi}{12}$ **B.** $\dfrac{\pi}{5}$ **C.** $\dfrac{4}{3}$ **D.** $\dfrac{\pi}{12}$

A15 The sequence generated by the recurrence $a_{n+1} = \sqrt{3 + 2a_n}$ for positive integers n and $a_1 = 1$ is known to be convergent. What is $\lim_{n\to\infty} a_n$?

A. -1 **B.** 0 **C.** $\sqrt{5}$ **D.** 3

A16 The series $1 + \dfrac{x^2}{1!} + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \cdots + \dfrac{x^{2n}}{n!} + \ldots$ is the Maclaurin series expansion of

A. $\cos x$ **B.** $\cos(x^2)$ **C.** e^x **D.** e^{x^2}

A17 The infinite series $1 + 2x + 2^2x^2 + 2^3x^3 + \ldots$ has radius of convergence

A. 1 **B.** 2 **C.** -2 **D.** $\dfrac{1}{2}$

A18 $\dfrac{d}{dx}\displaystyle\int_0^x \sin(t^{100})\,dt =$

A. $x^{101}\cos(x^{100})$ **B.** $\dfrac{x^{101}\cos x}{101}$ **C.** $\sin(x^{101})$ **D.** $\sin(x^{100})$

A19 The differential equation $\dfrac{dy}{dx} + P(x)y = Q(x)$ can be solved by multiplying with the *integrating factor*

A. $\displaystyle\int P(x)\,dx$ **B.** $\displaystyle\int Q(x)\,dx$ **C.** $e^{\int P(x)\,dx}$ **D.** $e^{\int Q(x)\,dx}$

A20 The function $f : \mathbb{R} \to \mathbb{R}$ is defined by the rule $f(x) = \dfrac{a\sin x}{x}$ when $x \neq 0$ and $f(0) = 3$. If f is given to be continuous then $a =$

A. 0 **B.** 1 **C.** 3 **D.** 4

A21 Let $h(x) = f(x)^2 + g(x)^2$ where $f(x)$ and $g(x)$ satisfy the relations

$$\frac{d}{dx}f(x) = g(x) \quad \text{and} \quad \frac{d}{dx}g(x) = -f(x).$$

Then $\dfrac{dh}{dx} =$

A. 0 **B.** 1 **C.** 2 **D.** insufficient data

A22 How many complex numbers z satisfy the relations $|z-1| = 1$ and $|z| = |z-i|$ simultaneously? (You might find it helpful to sketch the points on an Argand diagram.)

A. 0 **B.** 1 **C.** 2 **D.** 4

Section B

B1 Use induction to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \in \mathbb{N}$. ***(5 marks)***

B2 Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = \cos t$$

where y is a function of t. ***(5 marks)***

B3 Write down the expression for $f'(x)$, the derivative of a function f at x, as a suitable limit. Using this, show that if $f(x) := \frac{1}{x}$ then $f'(x) = -\frac{1}{x^2}$. ***(3 marks)***

B4 Sketch a picture to illustrate why

$$(b-a)a^2 \leq \int_a^b x^2\,dx \leq (b-a)b^2$$

for $0 \leq a < b$.

Using the above estimates and a division of the interval $[0,1]$ into N congruent subintervals, obtain *underestimates* and *overestimates* for the area $\int_0^1 x^2\,dx$. Deduce that

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

You may use the result of Question **B1** without proof if needed. ***(5 marks)***

B5 Write down the coefficient of x^n in the binomial expansion of $(1+x)^a$ for a general exponent a. Then use the relation

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

to write down the first three non-zero terms in the Maclaurin series expansion of $\arcsin(x)$. Hence evaluate

$$\lim_{x\to 0}\frac{6\arcsin(x) - 6x - x^3}{x^5}.$$

(You do not have to worry about the radii of convergence of the series involved.)

(5 marks)

B6 Evaluate $\displaystyle\int_a^\infty \frac{1}{x^3}\,dx$ when $a > 0$.

Assume that the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$ converges to ℓ. How many terms of the series should you add up in order to determine ℓ to within an error of $\dfrac{1}{2,0000,000}$?

(5 marks)

B7 Express $1 + i\sqrt{3}$ in polar form. By considering $(1+i\sqrt{3})^n$ in two ways, show that

$$1 - \binom{n}{2}3 + \binom{n}{4}3^2 - \binom{n}{6}3^3 + \cdots = 2^n \cos\frac{n\pi}{3}$$

for any positive integer n. Also, find a closed form expression for

$$\binom{n}{1} - \binom{n}{3}3 + \binom{n}{5}3^2 - \binom{n}{7}3^3 + \ldots.$$

(5 marks)

End of Question Paper