The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn 2017**

Advanced Calculus and Linear Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper E denotes an identity matrix.

1 (i) Let $(x, y, z) = F(u, v, w)$ be a smooth map defined for $(u, v, w) \in \mathbb{R}^3$ and let $s = G(x, y, z)$ be a smooth map such that $(G \circ F)(u, v, w)$ is defined for all $(u, v, w) \in \mathbb{R}^3$.

(a) State the Chain Rule for $G \circ F$ as a matrix equation.

(b) Now write down the equation for $\dfrac{\partial s}{\partial u}$ in terms of partial derivatives of s and partial derivatives of x, y, z. **(5 marks)**

(ii) Let $(x, y, z) = F(u, v, w)$ be the smooth map

$$F(u, v, w) = (u + v + w, uv + vw + wu, uvw)$$

for $(u, v, w) \in \mathbb{R}^3$.

(a) Calculate the derivative matrix $D(F)(u, v, w)$.

(b) Calculate the Jacobian of F, expressing your answer as a product of three factors.

(c) Find the rank and nullity of $D(F)(u, v, w)$ at each $(u, v, w) \in \mathbb{R}^3$. State clearly any general result about rank, nullity and determinants which you use. **(10 marks)**

(iii) Let V be the nullspace of a non-zero linear map $L\colon \mathbb{R}^p \to \mathbb{R}$.

(a) Explain why $\dim V = p - 1$.

(b) Let $\mathbf{b}_1, \ldots, \mathbf{b}_{p-1}$ be a basis for V, and let $\mathbf{y}$ be an element of $\mathbb{R}^p$ which is not in V.

Show that $\mathbf{b}_1, \ldots, \mathbf{b}_{p-1}, \mathbf{y}$ is a basis for $\mathbb{R}^p$.

(c) Now let V be the subspace of $\mathbb{R}^4$ defined by $x - 2y + 3z - 5t = 0$.

Find a basis for V and extend it to a basis for $\mathbb{R}^4$. **(10 marks)**

2 (i) Evaluate the triple integral

$$\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=y^2}^{1} 12yze^{zx^2}\,dx\,dy\,dz.$$

(9 marks)

(ii) Let R be the region above the (x,y)-plane and below the surface $z = 2 - 4x^2 - y^2$.

(a) By using a double integral, or otherwise, calculate the volume of R.

(b) Let $T(x,y,z) = z$ be a temperature distribution defined for (x,y,z) in R. By using a modification of cylindrical coordinates, or otherwise, find the average of T over R. **(8 marks)**

(iii) Consider the smooth map

$$(x,y) = F(u,v) = (\ln(u^2v),\, uv), \qquad u > 0,\ v > 0.$$

(a) Fix $u_0 > 0$. Find the coordinate curve for $u = u_0$, expressing your answer as an intrinsic equation for x and y.

Now fix $v_0 > 0$. Find the coordinate curve for $v = v_0$, expressing your answer as an intrinsic equation for x and y.

(b) Let R' be the square $1 \leqslant u \leqslant 2$, $1 \leqslant v \leqslant 2$ in the (u,v)-plane. Write R for the set of points $(x,y) = F(u,v)$ for (u,v) in R'. Find the area of R. **(8 marks)**

3 (i) Find the eigenvalues of the matrix $A = \begin{bmatrix} 8 & 2 & 2 \\ 2 & -4 & 5 \\ 2 & 5 & -4 \end{bmatrix}$. **(6 marks)**

(ii) Find an orthonormal set of eigenvectors corresponding to the eigenvalues found in (i). **(9 marks)**

(iii) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. **(4 marks)**

(iv) Determine the canonical form of the quadratic form

$$Q(\mathbf{x}) = 8x_1^2 - 4x_2^2 - 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 10x_2x_3,$$

where $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$. What are the rank and signature of $Q(\mathbf{x})$? **(6 marks)**

4 (i) Evaluate the line integral

$$I = \int_{\mathcal{C}} (xy + z^3)\ \mathrm{d}s,$$

from $(1, 0, 0)$ to $(1, 0, 2\pi)$ along the curve $\mathcal{C}$ that is represented by the parametric equations

$$x = \cos 2t, \quad y = \sin 2t, \quad z = t, \qquad 0 \leqslant t \leqslant 2\pi.$$

(6 marks)

(ii) If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector field, state the relationships that hold between the partial derivatives of v_1, v_2 and v_3 if $\mathbf{v}$ is a conservative field.
Find the function $f(z)$ such that the vector field $\mathbf{v} = (xe^{-r^2}, ye^{-r^2}, f(z)e^{-r^2})$ is conservative, where $r^2 = x^2 + y^2 + z^2$. **(7 marks)**

(iii) Find a potential function $\Phi(x, y, z)$ for the vector field $\mathbf{v}$ in (ii).
(7 marks)

(iv) Show that the level surfaces $L_c(\Phi)$ corresponding to the values $c \in \mathbb{R}$ for the potential function that you found in (iii) are spheres centred on the origin. For what values c do these surfaces exist?
Using this result, or otherwise, explain why

$$I = \int_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\mathbf{r} = 0$$

for the vector field $\mathbf{v}$ in (ii), where $\mathcal{C}$ is any arc of a circle $\{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = a^2,\ z = 0\}$, for $a \in \mathbb{R}$. **(5 marks)**

End of Question Paper