Data provided: formula sheet

The University Of Sheffield.

MAS241

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2017–18

Mathematics II (Electrical)

2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) (a) Let $f(t)$ be the function defined for $t \geq 0$ by

$$f(t) \ := \ \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Show that the Laplace transforms of $f(t)$ is given by

$$\mathcal{L}\{f(t)\} \ = \ \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

(4 marks)

(b) Consider the differential equation

$$y''(t) + 4y(t) \ = \ f(t)$$

with initial conditions $y(0) = y'(0) = 0$, where $f(t)$ is the function defined in Part (a). Show that the Laplace transform of $y(t)$ is given by

$$Y(s) \ = \ \frac{1}{s^2(s^2+4)} - \frac{e^{s}}{s^2(s^2+4)}.$$

(4 marks)

(c) Determine the function $y(t)$ from Part (b) that satisfies the differential equation

$$y''(t) + 4y(t) \ = \ f(t)$$

with initial conditions $y(0) = 0$, $y'(0) = 0$. **(8 marks)**

(ii) Write down the formula for the convolution of two functions and find an explicit formula for $g * k(t)$ when $g(t) = H(t)t^n$ and $k(t) = H(t-1)$. **(4 marks)**

2 Consider the periodic function $f(t)$ with fundamental period $T = 2\pi$ defined on $[-\pi, \pi)$ by $f(t) = \pi - |t|$.

(i) Find the Fourier series $S[f](t)$ of $f(t)$. **(12 marks)**

(ii) Sketch the graph of the Fourier series you calculated in Part (a) over the interval $[-2\pi, 3\pi]$. **(3 marks)**

(iii) Write down the exponential form of the Fourier series of $f(t)$. **(3 marks)**

(iv) What is the exact value of $S[f]\left(\frac{101\pi}{2}\right)$? **(2 marks)**

3 (i) Let $f(x,y) = x\sin(xy)$. Calculate the partial derivatives

$$f_x,\ f_y,\ f_{xx},\ f_{yx},\ f_{xy}.$$

(5 marks)

(ii) Find and classify *all* the critical points of the function

$$g(x,y) \;=\; -\frac{x^4}{4} + \frac{2x^3}{3} + 4xy - y^2.$$

(12 marks)

(iii) Let $p(x,y)$ be a polynomial with $p(0,y) = y$, $p(x,0) = x$ and $p_{yx} = 8x^3y$. Find $p(x,y)$. **(3 marks)**

4 (i) Let $R = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le 2\}$. Calculate

$$\iint_R x^3 e^y \; dA.$$

(7 marks)

(ii) Let P be the region of $\mathbb{R}^3$ above the square in the xy-plane with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ and below the plane defined by $x + z = 1$. Find

$$\iiint_P x^3y^2z \; dV.$$

(7 marks)

(iii) Calculate

$$\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=1} e^{x^3} \; dxdy.$$

(6 marks)

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} \;=\; \left(e^{xyz},\ \frac{1}{xyz},\ x + y^2 + z^3\right).$$

(a) Calculate **div F**. **(4 marks)**

(b) Calculate **curl F**. **(4 marks)**

(ii) Let $f(x, y) = \sqrt{6 - x^2 - y^2}$.

(a) Calculate the directional derivative of $f(x, y)$ at $(x, y) = (1, 1)$ in the $\mathbf{v} = (3, -4)$ direction. **(4 marks)**

(b) In which direction is $f(x, y)$ most rapidly increasing at the point $(x, y) = (-1, -1)$, and what is the maximum rate of increase? **(4 marks)**

(iii) Let $\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by $\mathbf{r} = (x, y, z)$, let $r : \mathbb{R}^3 \to \mathbb{R}$ be the scalar field given by $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Show that

$$\nabla g(r) \;=\; r^{-1}\frac{dg}{dr}\mathbf{r}.$$

(4 marks)

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad (\text{for } a > 0)$	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad (\text{for } a > 0)$	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \qquad (\text{for } n \geq 0)$	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T) \qquad (\text{for } T \geq 0)$	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T) \qquad (\text{for } T \geq 0)$	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r,\theta,z) = \left(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\right)$$
$$dV = r\,dr\,d\theta\,dz.$$

Spherical polar coordinates

$$(x,y,z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho,\theta,\phi) = \left(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\right)$$
$$dV = \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta.$$