Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.
The paper is marked out of a total of 60 marks.

1 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - 9\frac{\partial^2 y}{\partial x^2} = 0$$

has solutions of the form $y(x,t) = f(x + \lambda t)$ for arbitrary twice differentiable functions f, provided that $\lambda = -3$ or $\lambda = 3$.

Write down the general solution of the partial differential equation. ***(5 marks)***

Find the particular solution for y that satisfies the conditions

$$y(x,0) = \ln(1 + x^2)$$

and

$$\frac{\partial y}{\partial t}(x,0) = 2.$$

(10 marks)

2 (i) The power, P, of an engine depends on two parameters, x and y, through the formula

$$P = \sqrt{\frac{x^3}{y}}.$$

Under ideal working conditions, $x = 18$ and $y = 10$ (in arbitrary units). The quantity x is subjected to an increase of 2%. Using the formula for small increments, calculate the approximate percentage change needed in y to ensure that the power of the engine does not change. ***(7 marks)***

(ii) Show that if g is a differentiable real function and if $z = g\left(\frac{x}{y}\right)$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

(3 marks)

Verify that this equation is satisfied by the function

$$z = \exp\left(\frac{x-y}{x+y}\right).$$

(5 marks)

3 (i) Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F} = (xz, (3x+2y)^3, \exp(x+y+z)).$$

(a) Calculate $\nabla \cdot \mathbf{F}$. ***(2 marks)***

(b) Calculate $\nabla \times \mathbf{F}$. ***(3 marks)***

(ii) A continuous random variable X has probability density function

$$p(x) = \begin{cases} ce^{-x} & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

(a) Find the value of the constant c. ***(1 mark)***

(b) Find the cumulative distribution function (CDF) of X. ***(3 marks)***

(c) Find $P(1 < X < 3)$, where P denotes the probability. ***(2 marks)***

(iii) Show that the function $f(x) = x^6 - x - 1$ has a root in the interval $[1, 2]$. Write down the iteration formula for the Newton-Raphson method. Starting with an initial guess of $x_0 = 1.5$, use the Newton-Raphson method to find the root of $f(x)$ that is in the vicinity of 1.5 correct to 3 decimal places. ***(4 marks)***

4 (i) A periodic function, $f(t)$, with period 2π is defined by

$$f(t) = t^3 \quad \text{for} \quad -\pi \leq t < \pi.$$

(a) Sketch a graph of the function $f(t)$ for values of t from $t = -4\pi$ to $t = 4\pi$. ***(2 marks)***

(b) Show that the Fourier series for $f(t)$ in the interval $-\pi \leq t < \pi$ is given by

$$f(t) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{6}{n^2} - \pi^2\right)\sin nt.$$

(6 marks)

(ii) The Dirichlet conditions are sufficient conditions for a real-valued, periodic function $h(x)$ to be equal to the sum of its Fourier series at each point where it is continuous. State two of the Dirichlet conditions. ***(2 marks)***

For each of the following functions state whether or not the Dirichlet conditions are satisfied. For those cases where the Dirichlet conditions are not satisifed, give a brief explanation of why that is the case.

(a) A periodic function, $g(x)$, of period 2π, defined by

$$g(x) = \frac{1}{x^2 - 1} \quad \text{for} \quad -\pi \leq x < \pi.$$

(b) A periodic function, $p(x)$, of period 6π, defined by

$$p(x) = \frac{x}{1 + x^2} \quad \text{for} \quad -3\pi \leq x < 3\pi.$$

(c) A periodic function, $r(x)$, of period π, defined by

$$r(x) = \sin\left(\frac{1}{2x - 1}\right) \quad \text{for} \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2}.$$

(5 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \qquad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$