Data provided: Formula Sheet

MAS250

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2017–18

Mathematics II (Materials) — 2 hours

Marks will be awarded for answers to all questions in Section A, and for your best THREE answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.
The maximum possible mark for the paper is 100.

Section A

A1 Find the general solution of the equation

$$\frac{1}{\cos x}\frac{dy}{dx}+y^3=0 \qquad \left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right).$$ ***(5 marks)***

A2 Find the particular solution of the equation

$$3\frac{d^2y}{dx^2}-5\frac{dy}{dx}-2y=7e^{2x}$$

which satisfies $\dfrac{dy}{dx}=0$ and $y=1$ when $x=0$. ***(13 marks)***

A3 A function $f=f(x,y)$ is a function of two variables x and y, which are given in terms of two other variables r and θ by

$$x=r\,e^{\theta} \qquad \text{and} \qquad y=r\,e^{-\theta}.$$

Using the chain rule, show that

$$2x\frac{\partial f}{\partial x}=r\frac{\partial f}{\partial r}+\frac{\partial f}{\partial \theta} \qquad \text{and} \qquad 2y\frac{\partial f}{\partial y}=r\frac{\partial f}{\partial r}-\frac{\partial f}{\partial \theta}.$$ ***(9 marks)***

A4 (a) Find a vector normal to the surface $\phi = -16$ at the point A with coordinates $(2, -1, 3)$, where

$$\phi = xe^{1+y} + y \sin \pi z - xz^2.$$ ***(5 marks)***

Find also the directional derivative of ϕ at A, in the direction $\mathbf{d} = (-1, 2, 1)$. ***(3 marks)***

(b) A vector field $\mathbf{u}$ is given for $x > 0$ by

$$\mathbf{u} = (\ln x, x^2yz, z^2 + y \cos z).$$

For $x > 0$ find $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$. ***(5 marks)***

Section B

B1 (a) Find the particular solution of the equation

$$(1 + x^2)\frac{dy}{dx} + 2xy = x \sin x$$

which satisfies $y = 1$ when $x = 0$. ***(9 marks)***

(b) Find the general solution of the equation

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = x + \sin x.$$ ***(11 marks)***

B2 (a) The following table shows the MAS110 marks ($= y$) and the UCAS points scores ($= x$) of 10 students:

x	144	216	136	136	164	140	168	160	172	184
y	68	67	60	18	88	60	62	73	62	52

Showing your working, calculate the mean and standard deviation of x and of y, and also the correlation between x and y. Give all your answers correct to three significant figures. ***(11 marks)***

Comment briefly on the implications of the correlation between x and y. ***(1 mark)***

(b) A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = (yz + xe^y, xz + y \cos z, \sinh(xy) - y).$$

Verify that

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0.$$ ***(8 marks)***

B3 A function $f(x)$ is defined on the interval $-1 \leqslant x \leqslant 1$ by

$$f(x) = \begin{cases} x & -1 \leqslant x \leqslant 0 \\ -x & 0 < x \leqslant 1. \end{cases}$$

(a) Show that $f(x)$ can be represented by the Fourier series

$$-\frac{1}{2}+\frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos(2m+1)\pi x}{(2m+1)^2}.$$ ***(15 marks)***

(b) Use the result of part (a) to find

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$ ***(5 marks)***

B4 The function $\phi(x,y)$ satisfies Laplace's equation in two dimensions, i.e.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

in the rectangular region $-1 < x < 0, \quad 0 < y < 2$. The boundary conditions are

$$\begin{aligned} \phi(0,y) &= 0 \\ \phi(x,0) &= 0 \\ \phi(x,2) &= 0 \\ \phi(-1,y) &= 1. \end{aligned}$$

If $\phi(x,y) = X(x)Y(y)$ for some functions X and Y, show that

$$\frac{X''}{X} = -\frac{Y''}{Y} = \alpha,$$

where α must be a constant. ***(3 marks)***

Find the values of $X(0)$, $Y(0)$ and $Y(2)$. ***(3 marks)***

Given that $\alpha > 0$, deduce that

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{2}\sin\frac{n\pi y}{2}$$

for some constants B_n. ***(8 marks)***

Show that

$$B_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{-4}{n\pi\sinh\frac{n\pi}{2}} & n \text{ odd.} \end{cases}$$ ***(6 marks)***

End of Question Paper

FORMULA SHEET

Trigonometry

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$, where $R = \sqrt{(a^2 + b^2)}$, $\cos\alpha = a/R$ and $\sin\alpha = b/R$

Hyperbolic Functions

$\sinh x = \dfrac{1}{2}(e^x - e^{-x})$

$\cosh x = \dfrac{1}{2}(e^x + e^{-x})$

$\cosh^2 x - \sinh^2 x = 1$

$\text{sech}^2 x + \tanh^2 x = 1$

$2\sinh x \cosh x = \sinh 2x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$

$\sinh^{-1} x = \ln\left[x + \sqrt{(1 + x^2)}\right]$, all x

$\cosh^{-1} x = \ln\left[x + \sqrt{(x^2 - 1)}\right]$, $x \geqslant 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$, $|x| < 1$

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |\, x \,| > 1$$

Differentiation and Integration

Function	Derivative
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec} x$	$-\text{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\coth x$	$-\text{cosech}^2 x$
$\text{sech} x$	$-\text{sech} x \tanh x$
$\text{cosech} x$	$-\text{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{(x^2+1)}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{(x^2-1)}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}, \quad \mid x \mid < 1$
$\coth^{-1} x$	$-\frac{1}{x^2-1}, \quad \mid x \mid > 1$

Function	Integral
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\left(\dfrac{x}{a}\right)$

Differentiation and Integration Formulae

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int_a^b uv\,dx = [u \times (\text{integral of } v)]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v)\,dx$$

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that x and y are functions of t, i.e., $x = x(t)$, $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

2. Suppose that $z = f(x, y)$ and that x and y are functions of the variables r and s, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

First-Order Differential Equations

1. **Direct Integration**

$$\frac{dy}{dx} = f(x)$$

$$y = \int f(x)dx + C$$

2. **Separation of Variables**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

3. **Homogeneous Equations**

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

make the substitution $y = zx$ to give

$$z + x\frac{dz}{dx} = f(z)$$

4. **Linear Equations**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply both sides by the integrating factor $e^{\int P(x)dx}$ to give

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

The Second-Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where a, b, and c are constants.

General solution is

$$y = \text{Complementary Function} + \text{Particular Integral}$$

The solution, y_c, is given by

(i) $y_c = Ae^{m_1 x} + Be^{m_2 x}$, if m_1 and m_2 real and different,

(ii) $y_c = e^{mx}(A + Bx)$, if m_1 and m_2 real and equal ($m_1 = m_2 = m$),

(iii) $y_c = e^{px}(A\cos qx + B\sin qx)$, if m_1 and m_2 are complex ($m_1 = p+iq$, $m_2 = p-iq$), where m_1 and m_2 are the roots of the *auxiliary equation*

$$am^2 + bm + c = 0$$

Particular Integral, y_p

$f(x) = Ax^2 + Bx + C$ $\quad\quad$ $y_p = ax^2 + bx + c$

$f(x) = Ae^{kx}$ $\quad\quad$ $y_p = ae^{kx}$
when k is not one of the roots of the auxiliary equation

$f(x) = Ae^{kx}$ $\quad\quad$ $y_p = axe^{kx}$
when k is one of the roots of the auxiliary equation

$f(x) = A\cos mx + B\sin mx$ $\quad\quad$ $y_p = a\cos mx + b\sin mx$
when $\sin mx$ or $\cos mx$ is not part of the complementary function

$f(x) = A\cos mx + B\sin mx$ $\quad\quad$ $y_p = x(a\cos mx + b\sin mx)$
when $\sin mx$ or $\cos mx$ is part of the complementary function

Fourier Series

Suppose that $f(x)$ is defined on the interval $-l \leqslant x \leqslant l$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) ,$$

where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx , \quad n = 0, 1, 2, \ldots ,$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx , \qquad n = 1, 2, \ldots .$$

On the interval $0 \leqslant x \leqslant l$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} , \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} , \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx .$$

Vector Calculus

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right) .$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Statistics

For data values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Means $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ etc.

Variances $s_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2) - \bar{x}^2$ etc.

s_x is standard deviation

Covariance $\mathrm{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n}\sum_{i=1}^{n}(x_i y_i) - \bar{x}\bar{y}$

Correlation coefficient $r = \frac{\mathrm{cov}(x, y)}{s_x s_y}$

Linear regression by least squares

The least squares fit to the linear relationship

$$y = a + b(x - \bar{x})$$

is given by

$$a = \bar{y}, \quad b = \frac{\mathrm{cov}(x, y)}{s_x^2}$$

The corresponding mean square residual is $s_y^2(1 - r^2)$.