Data provided: Formula sheet

The University Of Sheffield.

MAS252

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find the Fourier series representation of the function

$$f(x) = \begin{cases} 0, & \text{for } \; -\pi \le x \le 0 \\ x, & \text{for } \; 0 \le x \le \pi \end{cases}$$

in the interval $-\pi \le x \le \pi$. Discuss separately the values of the coefficients of the series for even or odd indices. For a suitable value of x, prove the identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

(16 marks)

(ii) Explain why the equation

$$3x - \cos x - 1 = 0,$$

has a root in the interval $(0.55; 0.75)$. Perform *four* iterations of the bisection method to find this root. Work correct to *three* decimal places.

(9 marks)

2 If heat is generated at a constant rate throughout a bar of length $L = \pi$ with initial temperature $f(x) = \pi^2 - x^2$, and the ends at $x = 0$ and $x = \pi$ are kept at temperature 0 (i.e. $u(0, t) = u(\pi, t) = 0$), the heat conduction equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + H, \quad u = u(x, t), \quad c > 0,$$

where $H > 0$ is constant. Use the method of separation of variables to find the solution, $u(x, t)$, of the above heat conduction equation.

Hint: first set $u(x, t) = v(x, t) - Hx(x - \pi)/2c^2$. ***(25 marks)***

3 (i) A class of Bessel functions you have met in many engineering applications is given by the differential equation

$$x^2y'' + y' + (25 - x^2)y = 0, \quad y = y(x).$$

Find the first *four* non-zero terms in its series expansion solution if the solution of the differential equation satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1/2$. ***(13 marks)***

(ii) If $V = V(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial V}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}.$$

Furthermore, prove the identities

$$\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \theta}\right)^2 = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2,$$

and

$$r\frac{\partial V}{\partial r}\frac{\partial V}{\partial \theta} = xy\left[\left(\frac{\partial V}{\partial y}\right)^2 - \left(\frac{\partial V}{\partial x}\right)^2\right] + (x^2 - y^2)\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}.$$

(12 marks)

4 (i) A small engineering company is specialised in testing hydraulic dampers. A carriage carrying a variable mass is projected along a track of very low friction at a carefully controlled speed. At the end of the track the carriage hits the damper and the compression of the damper is measured by a special device. After several tests it turns out that the compression of the damper is described by the equation

$$x = \frac{U}{\lambda} - \frac{b}{\lambda^2 m}\ln\left(\frac{b + \lambda U m}{b}\right),$$

where m is the mass of the carriage, U is the impact speed of the carriage, $\lambda = k/m$ is the elastic damping rate of the damper (k is the elastic constant), and b is the frictional force encountered by the carriage. The optimal compression of the damper occurs when $m = 100$ kg, $U = 2$ m/s, $\lambda = 1.5\ \text{s}^{-1}$, and $b = 2.75$ kg m/s^2.

Use the formula for small increments to find the change in the compression length of the damper if *all* parameters are increased by 10%. Work correct to *three* decimal places. ***(15 marks)***

(ii) The finite difference form of the first and second order derivative of a function, $U(x, t)$, can be written as

$$\left(\frac{\partial U}{\partial t}\right)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{k}, \quad \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2},$$

where $U_{i,j} = U(ih, jk)$.

 Question 4 continued on next page

4 (continued)

(a) Show that the *explicit* approximation to the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 5U, \quad U = U(x,t), \tag{1}$$

is given in the usual notation as

$$U_{i,j+1} = rU_{i-1,j} + (1 - 2r - 5k)U_{i,j} + rU_{i+1,j},$$

where $r = k/h^2$ and k and h denote the time and spatial steps, respectively.

(b) If the initial condition and boundary conditions associated with the equation (1) are

$$U(x,0) = 1 + x(1-x), \quad 0 \leq x \leq 1,$$

$$U(0,t) = U(1,t) = 0, \quad t \geq 0,$$

use the explicit scheme with $h = 0.2$ and $k = 0.002$ to find the values of $U(x,t)$ at $t = 0.002$, working correct to three decimal places.

(10 marks)

End of Question Paper

Formula sheet

- **Trigonometric identities**

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

- **The local truncation error in the case of the 4th order Runge-Kutta method is given by**

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \le x \le l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \le x \le l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

- **the orthogonality of the sine function can be defined as**

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \textbf{if } m \neq n \\ L/2 & \textbf{if } m = n \end{cases}$$