The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2017–2018

Continuum Mechanics — 2 hours

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 (i) Write down in full the following expressions:

$$\text{(a)}\ \delta_{ij}x_iy_j \quad \text{(b)}\ t_j = n_iT_{ij} \quad \text{(c)}\ u_i = \varepsilon_{ijk}U_{jk}.$$

(6 marks)

(ii) You are given that the matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \mu \end{pmatrix},$$

is proper orthogonal. Determine λ and μ. ***(7 marks)***

(iii) The matix $\hat{\mathbf{A}}$ is the transformation matrix from the old to the new Cartesian coordinates. The matrix of tensor $\mathbf{T}$ in the old coordinates is given by

$$\hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & b \\ 0 & b & 6 \end{pmatrix}.$$

In the new coordinates it has a diagonal form. Determine b.

[You can use without proof the relation $\hat{\mathbf{T}}' = \hat{\mathbf{A}}\hat{\mathbf{T}}\hat{\mathbf{A}}^T$ between the matrices of tensor $\mathbf{T}$ in the old and new coordinates.] ***(6 marks)***

(iv) The matrix of tensor $\mathbf{U}$ is given by

$$\hat{\mathbf{U}} = \begin{pmatrix} f & \cos x_2 & \sin x_3 \\ \cos x_2 & g & 2x_2x_3 \\ \sin x_3 & 2x_2x_3 & h \end{pmatrix},$$

where f, g, and h are functions of x_1, x_2, and x_3. You are given that this tensor is divergent-free, $\nabla\cdot\mathbf{U} = 0$. You are also given that $f = 0$ at $x_1 = 0$, $g = 0$ at $x_2 = 0$, and $h = 0$ at $x_3 = 0$. Determine f, g, and h. ***(6 marks)***

2 (i) Derive the mass conservation equation in Eulerian variables,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0.$$

[You can use without proof the following Lemma: If $f(\boldsymbol{r})$ is a continuous function and $\int_V f(\boldsymbol{r})\, dV = 0$ for any volume V, then $f(\boldsymbol{r}) \equiv 0$.]

(11 marks)

(ii) At the initial moment of time, $t = 0$, a continuum occupies a sphere of radius R. It starts to expand preserving its spherical shape. The velocity of any point at the surface of the sphere is perpendicular to the surface and its magnitude is equal to $V \cosh^{-2}(t/T)$, where V and T are constant. Inside the sphere the velocity is also in the radial direction and its magnitude is proportional to the distance from the centre of the sphere.

(a) Find the radius of the sphere at time t.

[You can use the relation $(\tanh x)' = \cosh^{-2} x$.] ***(3 marks)***

(b) Find the dependence of the velocity on the distance from the centre of the sphere, r, at time t. ***(2 marks)***

(c) You are given that the density in the sphere has the form $\rho = rf(t)$, and $f(0) = \rho_0/R$. Use the mass conservation equation to determine $f(t)$. Calculate the limiting value of the density as $t \to \infty$.

[You can use that for any vector $\boldsymbol{u}$, that only depends on r in the spherical coordinates, $\nabla \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r}$.] ***(9 marks)***

3 (i) By considering the equilibrium of an infinitesimal cube with the sides parallel to the coordinate axes, derive the equilibrium equation in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0,$$

where T_{ij} are the Cartesian components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\boldsymbol{b}$. Draw a sketch clearly indicating the forces acting on the cube. ***(15 marks)***

(ii) The density of a continuous medium is defined by

$$\rho = \frac{\rho_0 l^2}{l^2 + x_1^2 + x_2^2 + x_3^2},$$

where ρ_0 and l are constants. The matrix of the stress tensor $\mathbf{T}$ in this medium is given by

$$\hat{\mathbf{T}} = \frac{f}{l^4}\begin{pmatrix} x_1^2 & x_1x_2 & 0 \\ x_1x_2 & x_2^2 & 0 \\ 0 & 0 & ax_3^2 \end{pmatrix},$$

where f and a are constant. You are given that the medium is in equilibrium under the action of body force $\boldsymbol{b}$.

(a) Determine $\boldsymbol{b}$. ***(5 marks)***

(b) You are given that $\boldsymbol{b}$ is a potential vector field meaning that $\nabla \times \boldsymbol{b} = 0$. Determine a. ***(5 marks)***

4 (i) The motion of an ideal fluid is called potential if the velocity can be written in the form $\boldsymbol{v} = \nabla\varphi$, φ being called the velocity potential. Use Euler's equation for an incompressible homogeneous fluid written in the Gromeka-Lamb form,

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \Pi\right),$$

where Π is the body force potential, to derive the Lagrange-Cauchy integral for fluid potential motion,

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\|\boldsymbol{v}\|^2 + \Pi = f(t),$$

where $f(t)$ is an arbitrary function. ***(6 marks)***

(ii) A rigid sphere is immersed in an ideal incompressible fluid of density ρ. At the initial time it starts to expand, so that its radius is $R(t)$. At large distance from the sphere the water pressure is $p_0 = \text{const}$. There is no body force.

(a) You can assume that the water motion cased by the expansion of the sphere is axisymmetric, and the water speed is zero at large distance from the sphere. Use the mass coservation in Eulerian variables to calculate the water velocity and determine its potential.

[You can use that $\nabla \cdot \boldsymbol{v} = \dfrac{1}{r^2}\dfrac{\partial(r^2 v)}{\partial r}$ for a vector $\boldsymbol{v} = v\boldsymbol{e}_r$, where r is the radial coordinate of the spherical coordinate system and $\boldsymbol{e}_r$ is the unit vector in the radial direction.] ***(12 marks)***

(b) Use the Lagrange-Cauchy integral to calculate the force acting on the surface of the sphere. ***(7 marks)***

5 (i) You are given that, in equilibrium, the stress tensor $\mathbf{T}$ satisfies the equation written in Cartesian coordinates x_1, x_2, x_3,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = 0, \qquad (*)$$

where T_{ij} are the Cartesian components of the stress tensor $\mathbf{T}$, ρ is the density, and b_i are the components of the body force $\mathbf{b}$. You are also given that, in linear elasticity, the Cartesian components of the stress tensor are given by

$$T_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (\dagger)$$

where u_i are the components of the displacement $\mathbf{u}$, and λ and μ are the Lamé constants. Show that, when $b_i = 0$ in the linear elasticity equation $(*)$ reduces to

$$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} = 0. \qquad (\ddagger)$$

(6 marks)

(ii) There is a cylindrical rod of radius R and length L made of elastic isotropic material. There are two forces of magnitude F stretching the rod that are applied to the flat boundaries of the rod. They are parallel to the rod axis and act in opposite directions. The forces are evenly distributed on the flat surfaces, meaning that the surface traction is the same at all points of the flat surfaces.

(a) You can assume that, in cylindrical coordinates r, ϕ, z with the z-axis coinciding with the cylinder axis and the origin at the middle of the rod, the displacement in the cylinder has the form $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$, where $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are the unit vectors in the r and z-direction respectively. Show that, for this particular form of $\boldsymbol{u}$, equation $(\ddagger)$ reduces to the system of two equations,

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right) = 0, \qquad \frac{\partial^2 u_z}{\partial z^2} = 0.$$

Assuming that the displacement at the coordinate origin is zero show that the general solution to this system of equations regular at $r = 0$ is given by $u_r = Ar$, $u_z = Bz$, where A and B are constant.

[You can use without proof that, for $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$, $\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u})$. The general expression for the divergence in cylindrical coordinates is $\nabla\cdot\boldsymbol{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$.]

(7 marks)

 Question 5 continued on next page

5 (continued)

(b) Using equation (†) show that

$$\mathbf{T} = [2A(\lambda+\mu)+\lambda B](\boldsymbol{e}_r\boldsymbol{e}_r+\boldsymbol{e}_\phi\boldsymbol{e}_\phi)+[2A\lambda+B(\lambda+2\mu)]\boldsymbol{e}_z\boldsymbol{e}_z,$$

where $\boldsymbol{e}_\phi$ is the unit vector in the ϕ-direction. Use this expression to calculate the surface traction at the flat surfaces of the rod and at its side surface. Then, using the condition of continuity of the surface traction at these two surfaces, and neglecting the air pressure, determine the constants A and B, and express the rod elongation l in terms of R, L, F, and the Lamé constants λ and μ.

[You can use without proof that, for $\boldsymbol{u} = u_r(r)\boldsymbol{e}_r + u_z(z)\boldsymbol{e}_z$,
$\nabla\boldsymbol{u} = \dfrac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \dfrac{u_r}{r}\boldsymbol{e}_\phi\boldsymbol{e}_\phi + \dfrac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z$.] ***(12 marks)***

End of Question Paper