The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

Introduction to Relativity **2 hours**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

1 (i) (a) State the two postulates of special relativity. ***(4 marks)***

(b) Define an inertial frame of reference.
Give one example of an inertial frame and one example of a non-inertial frame. ***(3 marks)***

(ii) The inertial frame $\tilde{R} : (c\tilde{t}, \tilde{x})$ is moving at a constant velocity v relative to the inertial frame $R : (ct, x)$, such that the two frames are related by the two-dimensional *Lorentz transformation*

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = L(v) \begin{pmatrix} ct \\ x \end{pmatrix}, \qquad L(v) = \gamma(v) \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix},$$

with $\gamma(v) = \left(1 - v^2/c^2\right)^{-1/2}$.

(a) The rapidity ρ is defined by $\tanh \rho = v/c$.
Draw a labelled sketch of v/c as a function of ρ. ***(3 marks)***

(b) Show that $\gamma(v) = \cosh \rho$.
Show that $L(v) = H(\rho)$ where

$$H(\rho) = \begin{pmatrix} \cosh \rho & -\sinh \rho \\ -\sinh \rho & \cosh \rho \end{pmatrix}.$$

(5 marks)

(c) Show that

$$H(\alpha)H(\beta) = H(\alpha + \beta).$$

(You may use hyperbolic addition formulae without proof.) ***(4 marks)***

(d) Using part (ii)(c), or otherwise, find the velocity w for the Lorentz transformation $L(w) = L(u)L(v)$. ***(3 marks)***

(iii) An inertial observer sees two rockets A and B fly past in opposite directions with uniform velocities $-3c/4$ and $+3c/4$, respectively. Find the velocity of rocket B as seen from the inertial frame of rocket A. ***(3 marks)***

2 Two inertial frames $R : (ct, x, y, z)$ and $\tilde{R} : (c\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ are related by the transformation

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad L = \frac{1}{2} \begin{pmatrix} 4 & -2\sqrt{3} & 0 & 0 \\ -3 & 2\sqrt{3} & 0 & -1 \\ 0 & 0 & 2 & 0 \\ -\sqrt{3} & 2 & 0 & \sqrt{3} \end{pmatrix}.$$

(i) Write down the 4×4 matrix which is the metric tensor g. ***(2 marks)***

(ii) Show that L is a Lorentz transformation. ***(10 marks)***

(iii) Show that L is proper and orthochronous. ***(6 marks)***

(iv) L may be written as the product $L = L_\theta L_v$, where L_θ is a rotation about the y axis and L_v is a boost in the x direction.
Write down expressions for L_θ and L_v.
Find the associated velocity v and angle θ and hence interpret L geometrically. ***(7 marks)***

3 Three events A, B and C in an inertial frame R have space-time coordinates (ct, x, y, z) given by

$$A : (0, 0, 0, 0), \qquad B : (0, 1, 0, 0), \qquad C : (1, \varphi, 0, 0),$$

where $\varphi = \frac{1}{2}(\sqrt{5} + 1) \approx 1.618$ and $\varphi^{-1} = \varphi - 1$.

(i) Write down the following four-vectors:
(a) from A to B, (b) from B to C, (c) from C to A.
Classify each four-vector as time-like, space-like or null; and, if appropriate, as future-pointing or past-pointing. ***(6 marks)***

(ii) A rod aligned with the x-axis is moving at a velocity v in the x-direction. One end passes through the events B and C. The other end passes through the event A.

(a) Show that the rod is moving at velocity $v = c/\varphi$. ***(3 marks)***

(b) Using an appropriate Lorentz transformation, or otherwise, show that the events A and C are simultaneous in $\tilde{R}$, the rest frame of the rod. ***(3 marks)***

(c) Find the length of the rod in $\tilde{R}$. ***(3 marks)***

(d) Why can't two observers at rest in R and $\tilde{R}$, respectively, agree on the length of the rod? Discuss the phenomenon of *length contraction.* ***(3 marks)***

(iii) Draw a space-time diagram showing the (ct, x) axes of R; the $(c\tilde{t}, \tilde{x})$ axes of $\tilde{R}$; the events A, B, C; and the worldlines of the two ends of the rod. ***(7 marks)***

4 (i) A particle has worldline $X(t) = (ct, \mathbf{x}(t))$ in an inertial frame R.

(a) Define the quantities *proper time* τ, *four-velocity* V and *four-acceleration* A for this particle. ***(3 marks)***

(b) Show that the four-velocity V of the particle is

$$V = (\gamma(v)c, \gamma(v)\mathbf{v})$$

where $\mathbf{v} = \dfrac{d\mathbf{x}}{dt}$, $\gamma(v) = \left(1 - v^2/c^2\right)^{-1/2}$ and $v^2 = \mathbf{v} \cdot \mathbf{v}$. ***(3 marks)***

(c) Define the Lorentz bracket $g(A, B)$ where A and B are four-vectors. Show that $g(V, V) = c^2$. Classify V as timelike, spacelike or null. ***(5 marks)***

(d) An observer with four-velocity U observes the particle with four-velocity V. Show that the observer measures the speed of the particle to be $\tilde{v}$ given by

$$\frac{\tilde{v}}{c} = \sqrt{1 - \frac{c^4}{\left(g(U, V)\right)^2}}$$

(6 marks)

(ii) An astronaut undergoing constant acceleration a in an inertial frame R has a displacement vector $X(\tau)$ where

$$X(\tau) = \left(\frac{c^2}{a}\sinh\rho, \frac{c^2}{a}\left(\cosh\rho - 1\right), 0, 0\right), \qquad \rho(\tau) = \frac{a\tau}{c}.$$

The astronaut accelerates with $a = 10\text{ms}^{-2}$ from rest in R towards a star S, which is also at rest in R, and which is 10 light years away in R.

(a) How long does the journey take according to the astronaut? ***(5 marks)***

(b) How long does the journey take according to an observer at rest in frame R? ***(3 marks)***

5 (i) (a) Define the *rest mass* m and *four-momentum* P of a particle.

(2 marks)

(b) Express P in terms of the particle energy E and three-momentum $\mathbf{p}$.
Show that

$$E^2 = p^2c^2 + m^2c^4.$$

where $p^2 = \mathbf{p} \cdot \mathbf{p}$. ***(4 marks)***

(ii) A particle of rest mass M is at rest in an inertial frame R when it splits into two identical particles of rest mass μ, each of which moves off with identical speed u.

Show that the particles move off in opposite directions, and that each particle has a rest mass μ of

$$\mu = \frac{1}{2}M\left(1 - \frac{u^2}{c^2}\right)^{1/2}.$$

(9 marks)

(iii) Two particles, each of rest mass m, are moving with speed v in opposite directions along the x-axis of an inertial frame R. The two particles collide, and two new particles are formed, each with rest mass $10m$.

(a) Show that $v \geq \dfrac{3\sqrt{11}}{10}c$.
(Hint: First find an inequality for $\gamma(v)$.) ***(7 marks)***

(b) Now assume that the new particles are at rest in R.
Find the energy E of each particle before the collision in terms of m and c only. ***(3 marks)***

End of Question Paper