The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2017–18

WAVES 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

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1 The one-dimensional wave equation for $\phi(x, t)$ is

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

(i) Show that the general solution for $\phi\,(x, t)$ is

$$\phi(x, t) = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

(11 marks)

(ii) Given that

$$\phi(x, 0) = \begin{cases} 0 & (-\infty < x \leq -a) \\ a + x & (-a \leq x \leq 0) \\ a - x & (0 \leq x \leq a) \\ 0 & (a \leq x < \infty), \end{cases}$$

and that $\dfrac{\partial\phi(x, 0)}{\partial t} = 0$ for all x, find $\phi(x, t)$ where $a > 0$.

(9 marks)

(iii) Sketch the graph of $\phi\,(x, t)$ against x when $ct = 2a$.

(5 marks)

2 The vibration of a string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The boundary conditions are that $u = 0$ at both $x = 0$ and $x = a$, and initially, i.e. at $t = 0$, $\partial u/\partial t = 0$.

(i) Using the *method of separation of variables*, show that this configuration has the general solution

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{a}x \cos\frac{n\pi}{a}ct.$$

(16 marks)

(ii) If initially $u = f(x)$, find an expression for A_n in terms of $f(x)$.

(6 marks)

(iii) Show that each term in the general solution can be expressed in terms of

$$\sin\left[\frac{n\pi}{a}(x + ct)\right] \quad \text{and} \quad \sin\left[\frac{n\pi}{a}(x - ct)\right],$$

and give a brief physical interpretation.

(3 marks)

3 Consider acoustic waves in a closed cuboidal box with sides a_1, a_2, a_3. The origin O of a Cartesian coordinate system (x_1, x_2, x_3) is taken at one corner of the box with the axes parallel to the sides of the box. The velocity potential ϕ satisfies

$$\frac{\partial^2\phi}{\partial {x_1}^2} + \frac{\partial^2\phi}{\partial {x_2}^2} + \frac{\partial^2\phi}{\partial {x_3}^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2},$$

where the constant c is the speed of sound. The boundary conditions on ϕ are:

$$\text{(a)}\ \frac{\partial\phi}{\partial x_1} = 0 \text{ at } x_1 = 0,\ x_1 = a_1; \quad \text{(b)}\ \frac{\partial\phi}{\partial x_2} = 0 \text{ at } x_2 = 0,\ x_2 = a_2;$$

$$\text{(c)}\ \frac{\partial\phi}{\partial x_3} = 0 \text{ at } x_3 = 0,\ x_3 = a_3.$$

(i) Seek separable solutions of the form

$$\phi = X_1(x_1)X_2(x_2)X_3(x_3)\cos\omega t,$$

where ω is a positive constant.

Show that X_i''/X_i must be constant for each i, and that these constants must all be non-positive. Here $'$ denotes differentiation.

Deduce that

$$\phi \propto \cos\left(\frac{n_1\pi x_1}{a_1}\right)\cos\left(\frac{n_2\pi x_2}{a_2}\right)\cos\left(\frac{n_3\pi x_3}{a_3}\right)\cos\omega t,$$

where the non-negative integers n_1, n_2, n_3 satisfy

$$\frac{n_1^2}{A_1^2} + \frac{n_2^2}{A_2^2} + \frac{n_3^2}{A_3^2} = 1 \text{ with } A_1 = \frac{a_1\omega}{\pi c},\ A_2 = \frac{a_2\omega}{\pi c},\ A_3 = \frac{a_3\omega}{\pi c}.$$

(18 marks)

(ii) Deduce that, for a large box, the number of different waves with angular frequency less than or equal to ω is approximately equal to

$$\frac{\omega^3}{6\pi^2c^3}a_1a_2a_3.$$

(7 marks)

[*HINT. Consider the number of integer triples* (n_1, n_2, n_3) *within the surface* $\frac{x_1^2}{A_1^2} + \frac{x_2^2}{A_2^2} + \frac{x_3^2}{A_3^2} = 1$, *and how this number relates to the volume bounded by the surface which is given to be* $\frac{4\pi}{3}A_1A_2A_3$.]

4 The equilibrium position of the free surface of a liquid of depth h is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the free surface and

$$\eta = a \sin kx \cos \omega t,$$

where a, k and ω are positive constants with $ka \ll 1$. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\text{(a)} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0; \qquad \text{(b)} \quad \frac{\partial \phi}{\partial z} = 0 \ \text{ at } \ z = -h;$$

$$\text{(c)} \quad \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \ \text{ at } \ z = 0; \quad \text{(d)} \quad \frac{\partial \phi}{\partial t} + g\eta = 0 \ \text{ at } \ z = 0.$$

(i) Explain briefly why each of (a), (b), (c) and (d) hold.

(8 marks)

(ii) Show that all conditions can be satisfied by taking

$$\phi = f(z) \sin kx \sin \omega t$$

for a suitable $f(z)$, which is to be found, and provided

$$\omega^2 = gk \tanh(kh).$$

(11 marks)

(iii) Determine the phase velocity c and the group velocity c_g in terms of g, k and h. Show that

$$\frac{c_g}{c} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right].$$

(6 marks)

5 In a model of traffic flow in the direction of Ox, the density of traffic at time t is $\rho(x, t)$, the speed of traffic of density ρ is $v = v(\rho)$, the flowrate $q(\rho) = \rho v(\rho)$, and $c(\rho) = q'(\rho)$.

(i) *Given* that $\rho_t + c\rho_x = 0$, show that $c_t + cc_x = 0$. If $\rho(x, 0) = f(x)$, deduce that in regions where $c(x, t)$ is continuously differentiable:

$$c = c\{f(\xi)\} = F(\xi) \ \text{ on straight lines } \ x = \xi + F(\xi)t.$$

(13 marks)

 Question 5 continued on next page

5 (continued)

(ii) A shock occurs with values of $(\rho, \;\; q = q(\rho), \;\; c = c(\rho))$ on the two sides of the shock equal to $(\rho_1, \;\; q_1, \;\; c_1)$ and $(\rho_2, \;\; q_2, \;\; c_2)$.

Given that the speed U of the shock satisfies

$$U = \frac{q_1 - q_2}{\rho_1 - \rho_2},$$

show that

$$U = \frac{1}{2}(c_1 + c_2)$$

in the following two cases:

(a) exactly when $q(\rho)$ is a quadratic function of ρ;

(b) approximately when the shock is weak, i.e. when $|\rho_2 - \rho_1| \ll \rho_1$ and $|\rho_2 - \rho_1| \ll \rho_2$.

(12 marks)

End of Question Paper