SCHOOL OF MATHEMATICS AND STATISTICS  
Autumn Semester 2017–18

WAVES  
2 hours

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

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Registration number from U-Card (9 digits) to be completed by student

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The one-dimensional wave equation for $\phi(x, t)$ is

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. $$

(i) Show that the general solution for $\phi(x, t)$ is

$$\phi(x, t) = f(x - ct) + g(x + ct),$$

where $f$ and $g$ are arbitrary functions. \hspace{1cm} (11 marks)

(ii) Given that

$$\phi(x, 0) = \begin{cases} 
0 & (-\infty < x \leq -a) \\
0 & (a \leq x < \infty),
\end{cases}$$

and that $\frac{\partial \phi(x, 0)}{\partial t} = 0$ for all $x$, find $\phi(x, t)$ where $a > 0$. \hspace{1cm} (9 marks)

(iii) Sketch the graph of $\phi(x, t)$ against $x$ when $ct = 2a$. \hspace{1cm} (5 marks)

The vibration of a string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. $$

The boundary conditions are that $u = 0$ at both $x = 0$ and $x = a$, and initially, i.e. at $t = 0$, $\partial u/\partial t = 0$.

(i) Using the method of separation of variables, show that this configuration has the general solution

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{a} \cos \frac{n\pi}{a} ct. $$

(16 marks)

(ii) If initially $u = f(x)$, find an expression for $A_n$ in terms of $f(x)$. \hspace{1cm} (6 marks)

(iii) Show that each term in the general solution can be expressed in terms of

$$\sin \left[ \frac{n\pi}{a} (x + ct) \right] \quad \text{and} \quad \sin \left[ \frac{n\pi}{a} (x - ct) \right],$$

and give a brief physical interpretation. \hspace{1cm} (3 marks)
Consider acoustic waves in a closed cuboidal box with sides $a_1$, $a_2$, $a_3$. The origin $O$ of a Cartesian coordinate system $(x_1, x_2, x_3)$ is taken at one corner of the box with the axes parallel to the sides of the box. The velocity potential $\phi$ satisfies
\[
\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2},
\]
where the constant $c$ is the speed of sound. The boundary conditions on $\phi$ are:

(a) $\frac{\partial \phi}{\partial x_1} = 0$ at $x_1 = 0$, $x_1 = a_1$;
(b) $\frac{\partial \phi}{\partial x_2} = 0$ at $x_2 = 0$, $x_2 = a_2$;
(c) $\frac{\partial \phi}{\partial x_3} = 0$ at $x_3 = 0$, $x_3 = a_3$.

(i) Seek separable solutions of the form
\[
\phi = X_1(x_1)X_2(x_2)X_3(x_3)\cos \omega t,
\]
where $\omega$ is a positive constant.

Show that $X_i''/X_i$ must be constant for each $i$, and that these constants must all be non-positive. Here $'$ denotes differentiation.

Deduce that
\[
\phi \propto \cos \left( \frac{n_1 \pi x_1}{a_1} \right) \cos \left( \frac{n_2 \pi x_2}{a_2} \right) \cos \left( \frac{n_3 \pi x_3}{a_3} \right) \cos \omega t,
\]
where the non-negative integers $n_1$, $n_2$, $n_3$ satisfy
\[
\frac{n_1^2}{A_1^2} + \frac{n_2^2}{A_2^2} + \frac{n_3^2}{A_3^2} = 1 \quad \text{with} \quad A_1 = \frac{a_1 \omega}{\pi c}, \quad A_2 = \frac{a_2 \omega}{\pi c}, \quad A_3 = \frac{a_3 \omega}{\pi c}.
\]

(18 marks)

(ii) Deduce that, for a large box, the number of different waves with angular frequency less than or equal to $\omega$ is approximately equal to
\[
\frac{\omega^3}{6 \pi^2 c^3} a_1 a_2 a_3.
\]

(7 marks)

[HINT. Consider the number of integer triples $(n_1, n_2, n_3)$ within the surface $\frac{x_1^2}{A_1^2} + \frac{x_2^2}{A_2^2} + \frac{x_3^2}{A_3^2} = 1$, and how this number relates to the volume bounded by the surface which is given to be $\frac{4 \pi}{3} A_1 A_2 A_3$.]
The equilibrium position of the free surface of a liquid of depth \( h \) is \( z = 0 \), where \( z \) is measured vertically upwards. A surface wave causes the displacement of this surface to be \( \eta(x, t) \), where \( x \) is measured along the free surface and

\[
\eta = a \sin kx \cos \omega t,
\]

where \( a, k \) and \( \omega \) are positive constants with \( ka \ll 1 \). You are given that the velocity potential \( \phi = \phi(x, z, t) \) satisfies

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0; \quad (a) \\
\frac{\partial \phi}{\partial z} &= 0 \text{ at } z = -h; \quad (b) \\
\frac{\partial \phi}{\partial z} &= \frac{\partial \eta}{\partial t} \text{ at } z = 0; \quad (c) \\
\frac{\partial \phi}{\partial t} + g \eta &= 0 \text{ at } z = 0. \quad (d)
\end{align*}
\]

(i) Explain briefly why each of (a), (b), (c) and (d) hold.

(ii) Show that all conditions can be satisfied by taking

\[
\phi = f(z) \sin kx \sin \omega t
\]

for a suitable \( f(z) \), which is to be found, and provided

\[
\omega^2 = gk \tanh(kh).
\]

(iii) Determine the phase velocity \( c \) and the group velocity \( c_g \) in terms of \( g, k \) and \( h \). Show that

\[
\frac{c_g}{c} = \frac{1}{2} \left[ 1 + \frac{2kh}{\sinh 2kh} \right].
\]

In a model of traffic flow in the direction of \( Ox \), the density of traffic at time \( t \) is \( \rho(x, t) \), the speed of traffic of density \( \rho \) is \( v = v(\rho) \), the flowrate \( q(\rho) = \rho v(\rho) \), and \( c(\rho) = q'(\rho) \).

(i) Given that \( \rho_t + c \rho_x = 0 \), show that \( c_t + c c_x = 0 \). If \( \rho(x, 0) = f(x) \), deduce that in regions where \( c(x, t) \) is continuously differentiable:

\[
c = c\{f(\xi)\} = F(\xi) \text{ on straight lines } x = \xi + F(\xi)t.
\]
(ii) A shock occurs with values of \((\rho, \ q = q(\rho), \ c = c(\rho))\) on the two sides of the shock equal to \((\rho_1, \ q_1, \ c_1)\) and \((\rho_2, \ q_2, \ c_2)\).

Given that the speed \(U\) of the shock satisfies

\[
U = \frac{q_1 - q_2}{\rho_1 - \rho_2},
\]

show that

\[
U = \frac{1}{2}(c_1 + c_2)
\]

in the following two cases:

(a) exactly when \(q(\rho)\) is a quadratic function of \(\rho\);

(b) approximately when the shock is weak, i.e. when \(|\rho_2 - \rho_1| \ll \rho_1\) and \(|\rho_2 - \rho_1| \ll \rho_2\).

\(12\) marks

End of Question Paper