The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017-2018**

Complex Analysis **2 hours 30 minutes**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1 (i) State, without proof, the triangle inequalities for $|z+w|$ and $|z-w|$. *(2 marks)*

(a) Show that for all z on the circle $|z|=1$,

$$\frac{1}{3} \le \left|\frac{1}{2z+1}\right| \le 1. \qquad \textit{(3 marks)}$$

(b) Show that, for all z on the circle $|z-2| = 3$,

$$\frac{1}{7} \le \left|\frac{1}{z+2}\right| \le 1. \qquad \textit{(4 marks)}$$

(ii) Find all the solutions of the equation

$$e^{2z} + 1 + \sqrt{3}\,i = 0. \qquad \textit{(3 marks)}$$

(iii) Find all the solutions of the equation

$$\cosh z = -3. \qquad \textit{(3 marks)}$$

(iv) The path γ consists of the straight line segment from 0 to i followed by the straight line segment from i to $1+i$. Thus γ is the path γ_1 followed by the path γ_2, where γ_1 is given by $z = it(0 \le t \le 1)$ and γ_2 is given by $z = t + i(0 \le t \le 1)$.

(a) Evaluate

$$\int_\gamma \bar{z}\,dz. \qquad \textit{(7 marks)}$$

(b) Evaluate

$$\int_\gamma (4z^3 \sinh z^2 + 2z^5 \cosh z^2)\,dz. \qquad \textit{(3 marks)}$$

2 (i) Define what is meant by the following three statements:

(a) D is a region in the complex plane;

(b) The function f is differentiable at a point z_0;

(c) The function f is analytic in the region D. ***(4 marks)***

State, without proof, the Cauchy-Riemann equations for a differentiable function. ***(2 marks)***

(ii) In each of the following cases, decide whether there is a function g analytic on $\mathbb{C}$ such that $\text{Re}[g(x+iy)] = u(x,y)$. When g exists find an explicit expression for $g(z)$ in terms of z

(a) $u(x,y) = y^2 - x^2$, ***(1 mark)***

(b) $u(x,y) = y^2 + x^2$, ***(1 mark)***

(c) $u(x,y) = \sin x \cosh y - 2\cos x \sinh y$. ***(7 marks)***

(iii) Let γ be the circular contour $|z| = 2$ described in the positive direction. Prove that

$$\left| \int_\gamma \frac{\bar{z}}{5+2iz}\, dz \right| \leq 8\pi.$$ ***(4 marks)***

(iv) Let $H = \{z \in \mathbb{C} : \text{Re} z > 0\}$. Show that

$$\int_\alpha \frac{\cosh z}{\sinh z}\, dz = \ln\left(\frac{e^2+1}{e}\right)$$ ***(6 marks)***

for all paths α in the half plane H which have initial point 1 and final point 2.

3 (i) State, without proof, Cauchy's Theorem and Cauchy's Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. ***(7 marks)***

(ii) Let γ be the circular contour $|z-2|=3$ described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(a) $$\int_\gamma z^4 \sin(\pi z^2)\, dz\,,$$

(b) $$\int_\gamma \frac{e^z}{z^2-16}\, dz\,,$$

(c) $$\int_\gamma \frac{\cosh 2z}{(z-1)^5}\, dz\,,$$

(d) $$\int_\gamma \mathrm{Im}(z)\, dz\,.$$

(18 marks)

4 (i) State without proof Liouville's Theorem. ***(2 marks)***

(ii) The function f is analytic in $\mathbb{C}$ and satisfies the relation $\mathrm{Im} f(z) \leq -3$ for all $z \in \mathbb{C}$. Show that f is constant. ***(5 marks)***

(iii) Explain how Laurent expansions are used to classify isolated singularities. ***(5 marks)***

(iv) Find the singularity of the following function in $\mathbb{C}$. Classify it and find the residue at the singularity.

$$z \sin\left(\frac{1}{z+1}\right)$$

(5 marks)

(v) For each of the following functions, find all the singularities in the complex plane. Classify them and find the Residues at each of them:

(a) $\dfrac{\cos \pi z}{e^z(2z+1)}$; (b) $\dfrac{e^z}{1+e^z}$.

(8 marks)

5 (i) State, without proof, Cauchy's Residue Theorem. Your statement should include conditions under which the result is valid. ***(4 marks)***

(ii) Let γ be the circular contour $|z| = 3$ described in the anti-clockwise direction. Evaluate

$$\int_\gamma \frac{(z+1)\,\exp(1/z)}{z}\,dz\ ,$$ ***(10 marks)***

(iii) Prove that

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+4)(x^2+9)}\,dx \;=\; \frac{2\pi}{5}\left(\frac{1}{4e^4} - \frac{1}{6e^6}\right)$$

(11 marks)

End of Question Paper