The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017-18**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Subfield Criterion. ***(4 marks)***

(ii) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{-2} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{2} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(iii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $a = \sqrt{2} - 2\sqrt{3}$.

(a) Show that $L = \mathbb{Q}(a)$. ***(5 marks)***

(b) Express the element a^{-1} as a sum $\lambda_0 + \lambda_1\sqrt{2} + \lambda_2\sqrt{3} + \lambda_3\sqrt{6}$ where $\lambda_i \in \mathbb{Q}$. ***(3 marks)***

(c) Express the element $b = 2a^2\sqrt{2} - 4a\sqrt{3} + 10a - 10$ as a sum $\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3$ where $\alpha_i \in \mathbb{Q}$. ***(5 marks)***

2 Let $K \subseteq L$ be a field extension.

(i) What is meant by saying that an element $a \in L$ is *algebraic* over K? ***(2 marks)***

(ii) What is meant by saying that the field extension L is a *simple* field extension of K? ***(2 marks)***

(iii) Give a definition of the *minimal polynomial* $m(x) \in K[x]$ of the algebraic element a over K and prove that it is an irreducible polynomial over K. ***(6 marks)***

(iv) Suppose that $n = \deg(m(x))$. Show that $[K(a) : K] = n$ and find a K-basis the vector space $K(a)$ over the field K. ***(9 marks)***

(v) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{3})[x]$ of the element $a = -\sqrt{2} + \sqrt{3}$ over the field $\mathbb{Q}(\sqrt{3})$. ***(6 marks)***

3 (i) Give a definition of the content $c(f)$ of a polynomial $f \in \mathbb{Z}[x]$. What does it mean to say that the polynomial f is primitive? ***(4 marks)***

(ii) Prove that $c(fg) = c(f)c(g)$ for all polynomials $f, g \in \mathbb{Z}[x]$. ***(5 marks)***

(iii) State and prove Gauss's Lemma. ***(11 marks)***

(iv) Let $f = g_1 g_2 \cdots g_m$ where $g_n = \sum_{i=n}^{2n} n^i x^i$ where $n = 1, 2, \ldots, m$. Find the content of the polynomial f. ***(5 marks)***

4 (i) Give the definition of a constructible number. ***(2 marks)***

(ii) Define a Fermat prime number. ***(2 marks)***

(iii) Let p be an odd prime number which is not a Fermat prime. Prove that the regular p-gon cannot be constructed. ***(8 marks)***

(iv) Prove that for all odd prime numbers p, the regular p^2-gon cannot be constructed. ***(7 marks)***

(v) Which of the following n-gons can be constructed $n = 60, 72, 165$? Justify your response. ***(6 marks)***

End of Question Paper