The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2017–18

Combinatorics — 2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) State Pascal's Identity. ***(2 marks)***

(b) State the Binomial Theorem. ***(2 marks)***

(c) By multiplying both sides of the equation appearing in the Binomial Theorem by $1 + x$, prove Pascal's Identity, ***(4 marks)***

(ii) (a) How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 19,$$

in which each x_i is a non-negative integer? Give a brief reason for your answer. ***(3 marks)***

(b) How many solutions are there as in part (a) such that $x_1 > 2$ or $x_2 > 3$ or $x_3 > 4$? ***(8 marks)***

(iii) Define a sequence of numbers s_n, for $n \geq 0$, by the recurrence relation

$$s_0 = 1,$$
$$s_n = 2s_{n-1} + 2^{n-1}.$$

Show that

$$s_n = \sum_{i=0}^{n} (i+1) \binom{n}{i}.$$

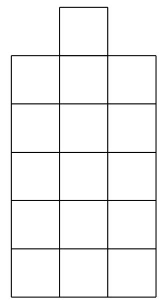
(6 marks)

2 (i) Let $n \geq 6$. Consider a rectangle n squares high and $n-3$ squares wide.

(a) Show that this can be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares).

(2 marks)

(b) Consider the same rectangle with the two top corner squares removed. (The case $n = 6$ is pictured below.)



Show that this can be completely covered by non-overlapping dominoes if and only if n is odd. ***(4 marks)***

(ii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Let X be a set of 12 numbers from $\{1, 2, \ldots, 100\}$. Show that there are two subsets of X each having exactly 5 elements and such that the sum of their elements is the same. ***(4 marks)***

(iii) (a) Consider the sets

$$A_1 = \{2, 4, 5\}, A_2 = \{1, 4, 5\}, A_3 = \{1, 6, 7\}, A_4 = \{2, 3, 6\}.$$

Can distinct representatives of these sets be chosen to include 3, 6 and 7 ? ***(1 mark)***

(b) State a necessary and sufficient condition for sets A_1, A_2, ..., A_n to have distinct representatives. ***(2 marks)***

(iv) Recall that a derangement of $\{1, 2, ..., n\}$ is a permutation leaving none of the numbers fixed. We write d_n for the number of derangements of $\{1, 2, ..., n\}$.

(a) Show that

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

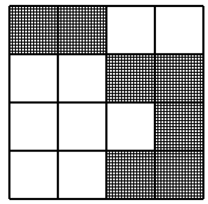
(4 marks)

(b) Show that, for $n \geq 3$,

$$d_n = (n-1)(d_{n-2} + d_{n-1}).$$

(6 marks)

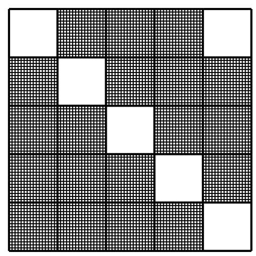
3 (i) Calculate the rook polynomial of the (unshaded) board B:



(6 marks)

(ii) Let $n \geq 3$.

(a) Let B_n be an $n \times n$ board where the only unshaded squares are those on the main diagonal top left to bottom right and the top right square. (The case $n = 5$ is pictured below.)



Show that the number of ways of placing k non-challenging rooks on B_n is

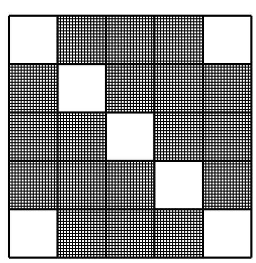
$$\binom{n}{k} + \binom{n-2}{k-1}.$$

(6 marks)

(b) Hence, or otherwise, show that the number of ways of placing k non-challenging rooks on C_n is

$$\binom{n}{k} + \binom{n-1}{k-1} + \binom{n-2}{k-1},$$

where C_n is the $n \times n$ board with the only unshaded squares being those on the main diagonal top left to bottom right, the top right square and the bottom left square. (Again the case $n = 5$ is pictured below.)



(6 marks)

 Question 3 continued on next page

3 (continued)

(iii) (a) Show that there is a tournament of 8 players with scores

$$6,\ 4,\ 4,\ 4,\ 4,\ 2,\ 2,\ 2.$$

(4 marks)

(b) Deduce that there is a a tournament of 16 players with scores

$$14,\ 12,\ 12,\ 12,\ 12,\ 10,\ 10,\ 10,\ 6,\ 4,\ 4,\ 4,\ 4,\ 2,\ 2,\ 2.$$

(3 marks)

4 (i) State necessary and sufficient conditions for a $p \times q$ Latin rectangle to be extendable to an $n \times n$ Latin square. ***(2 marks)***

(ii) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 1 & 6 & 5 \\ 6 & 3 & 5 & 4 \\ x & 5 & 4 & 1 \end{pmatrix}$$

Write down one such extension. ***(6 marks)***

(iii) Prove that there exist at most $n-1$ mutually orthogonal $n \times n$ Latin squares. ***(8 marks)***

(iv) Consider a (v, b, r, k, λ) design, where v is the number of varieties and b is the number of blocks. Explain the meanings of the other parameters r, k and λ. ***(3 marks)***

(v) Consider, as varieties, vectors of the form $\mathbf{x} = (x_1, x_2, x_3, x_4)$, where $x_1, x_2, x_3, x_4 \in \{0, 1\}$ and x_1, x_2, x_3, x_4 are not all zero. Given two different such vectors we add them using vector addition mod 2, that is,

$$(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (z_1, z_2, z_3, z_4),$$

where $z_i \in \{0, 1\}$ and $z_i \equiv x_i + y_i \pmod 2$ for $i = 1, 2, 3, 4$. Consider, as blocks, all sets of the form $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ such that $\mathbf{x} + \mathbf{y} + \mathbf{z} = (0, 0, 0, 0)$. Show that these are the blocks of a design and give all the parameters of the design. ***(6 marks)***

End of Question Paper