Data provided: list of formulae

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

Differential Geometry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

A list of formulae is provided on the last two pages.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Consider $b > a > 0$. Show that the function $\gamma_1(t) := (a\cos t, b\sin t)$, for $t \in \mathbb{R}$, parametrizes the set $\mathcal{E} := \{(x,y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. Sketch the parametrized curve γ_1 and prove that it is regular. ***(5 marks)***

Find the maximum and minimum values of the curvature of γ_1 and relate your answer to your sketch. ***(9 marks)***

(ii) Define what is meant by a unit-speed curve. Find a unit-speed reparametrization of the curve $\gamma_2 \colon (0,\infty) \to \mathbb{R}^2$, $t \mapsto (2\cos(t^2), 2\sin(t^2))$. ***(5 marks)***

Calculate a turning angle for your unit-speed curve and use this to calculate the curvature of the curve $\gamma_2(t)$. Comment on your answer. ***(6 marks)***

2 Suppose that a smooth, parametrized curve in $\mathbb{R}^3$ is defined by $\gamma(t) := (x(t), 0, z(t))$ for $t \in I$, where $I \subset \mathbb{R}$ is an open subset. Let σ be the parametrized surface of revolution around the z-axis. Give the parametrization of σ. Find necessary and sufficient conditions for this to be regular. ***(7 marks)***

Calculate the first fundamental form of this surface of revolution. Two of the entries are zero; state the geometric significance of this. ***(5 marks)***

Give the definition of a local isometry and state a necessary and sufficient condition on its first fundamental form for a parametrized surface to be a local isometry. Use this to show that the only parametrized surfaces of rotation which are local isometries are certain cylinders. ***(7 marks)***

Give the definition of a conformal parametrized surface and state a necessary and sufficient condition on its first fundamental form for a parametrized surface to be conformal. Use this to give one example of a parametrized surface of revolution which is conformal but not a local isometry, and one which is not conformal. ***(6 marks)***

3 Given a function $f\colon \mathbb{R}^2 \to \mathbb{R}$, consider the parametrized surface $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(u,v) := (u, v, f(u,v))$. What is the name of this parametrized surface? Under what conditions on f is this smooth and regular? Show that the Gaussian curvature of σ is as follows:

$$K(u,v) = \frac{f_{uu}(u,v)f_{vv}(u,v) - f_{uv}(u,v)^2}{(1 + f_u(u,v)^2 + f_v(u,v)^2)^2}.$$

(12 marks)

Now take $f(u,v) := uv$ and write σ^{uv} for the corresponding parametrized surface. State the Gaussian curvature of σ. Sketch σ^{uv} (or else describe carefully what it looks like). How does the shape of the surface tell you something about the sign of the Gaussian curvature at $\sigma^{uv}(0,0)$? What feature of the curve $t \mapsto \sigma^{uv}(0,t)$ allows you to know something about the sign of the Gaussian curvature at $\sigma^{uv}(0,v_0)$ for any v_0?

(7 marks)

The Weingarten matrix of σ^{uv} is as follows:

$$W = \frac{1}{\sqrt{1+u^2+v^2}^3}\begin{pmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{pmatrix}$$

Show that $\sqrt{1+u^2}\sigma^{uv}_u(u,v) + \sqrt{1+v^2}\sigma^{uv}_v(u,v)$ is a principal direction at $\sigma^{uv}(u,v)$ and find the corresponding principal curvature. At the point $\sigma^{uv}(0,0)$ how could you have deduced this was a principal direction by looking at the formula for σ^{uv} without any further calculation (or by looking at a sketch of the surface)? ***(6 marks)***

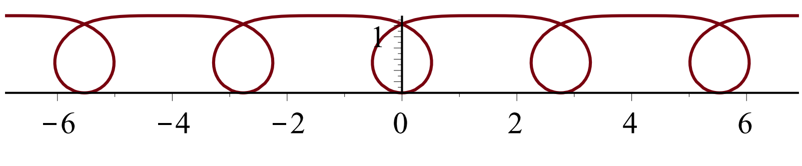
4 State whether each of the following is true or false and give a careful justification of your answer. Most of the marks are for the justification.

(i) There is a parametrized curve whose image is the set $\{(x,y) \mid xy = 1\}$.

(ii) The curve $\gamma_3\colon \mathbb{R} \to \mathbb{R}^2$, defined below is both smooth and regular:

$$\gamma_3(t) := \begin{cases} (t,0) & t \le 0, \\ (t,t^2) & t > 0. \end{cases}$$

(iii) The following is a picture of a smooth and regular parametrized curve $\gamma_4\colon\ ]-5\pi, 5\pi[\ \to \mathbb{R}^2$ which has its curvature function given by $\kappa(t) = \cos(t)$.



(iv) For a point $(x,y,z) \in \mathbb{R}^3$ in the image of a parametrized surface there is precisely one preferred normal to the surface at that point.

(v) There are maps of regions of the Earth which correctly represent the relative areas of countries.

(25 marks)

End of Question Paper

List of formulae

- The inverse of a 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients in $\mathbb{R}$ and $ad - bc \neq 0$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- The cross-product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ is

$$v_1 \times v_2 = (y_1 z_2 - z_1 y_2, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) \in \mathbb{R}^3.$$

- The angle θ between two vectors v_1 and $v_2 \in \mathbb{R}^3$ is given by

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}.$$

Inverse hyperbolic functions are given by the following.

- $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$
- $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$
- $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

The derivatives of the inverse hyperbolic functions are given by the following.

- $\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{x^2+1}}$
- $\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$
- $\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}$ for $|x| < 1$

For a curve on $\mathbb{R}^2$ parametrized by $\gamma \colon \,]\alpha, \beta[\, \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$:

- The arc length from $\gamma(a)$ to $\gamma(b)$, $\alpha < a \leq b < \beta$ is:

$$\int_a^b ||\dot{\gamma}(t)|| \, dt$$

- The curvature of γ at $\gamma(t)$ is

$$\kappa(t) = \frac{\ddot{\gamma}(t) \cdot J(\dot{\gamma}(t))}{\|\dot{\gamma}(t)\|^3} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the anti-clockwise rotation of angle $\pi/2$.

For a parametrized surface $\sigma \colon U \to \mathbb{R}^3$, with U an open set in $\mathbb{R}^2$:

- The first fundamental form is given by

$$\mathrm{I}_{(u,v)} = \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}$$

for all $(u, v) \in \mathbb{R}^2$, with $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$ and $G = \sigma_v \cdot \sigma_v$.

- Area of the domain $\sigma\left([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\right)$, for $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subseteq U$:

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \sqrt{EG - F^2}\, dv\, du$$

- The preferred unit normal vector along σ is given by $\mathbf{n} \colon U \to \mathbb{R}^3$,

$$\mathbf{n} = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||}.$$

- The second fundamental form of σ at $(u, v) \in U$ is

$$\mathrm{II}_{(u,v)} = \begin{pmatrix} L(u,v) & M(u,v) \\ M(u,v) & N(u,v) \end{pmatrix}$$

where $L = \sigma_{uu} \cdot \mathbf{n}$, $M = \sigma_{uv} \cdot \mathbf{n}$ and $N = \sigma_{vv} \cdot \mathbf{n}$.
- The Weingarten matrix of σ is

$$W = \mathrm{I}^{-1}\, \mathrm{II}\,.$$

- The Gaussian curvature is

$$K = \det W.$$

The Brioschi formula:

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}.$$