The dynamics of two interacting species, with densities $M$ and $N$, are governed by the following ordinary differential equations,

\[
\frac{dM}{dt} = M(a - bM + cN) \quad (1) \\
\frac{dN}{dt} = N(u - vN + wM), \quad (2)
\]

where $a, b, c, u, v, w$ are positive constants.

(i) Describe the type of interaction between these two species. \(\text{(2 marks)}\)

(ii) Besides the unstable equilibrium at $(0,0)$, this model yields three possible equilibria given by $(M_+, 0)$, $(0, N_+)$ and $(M^*, N^*)$.

(a) Draw a phase portrait for the case where all these equilibria exist in the biologically-feasible region (i.e. $M, N \geq 0$). This should clearly show the nullclines, equilibria, qualitative directions of flow and two example trajectories. \(\text{(5 marks)}\)

(b) Using the phase portrait, determine a condition on $w$ for the equilibrium $(M^*, N^*)$ to exist in the biologically-feasible region. \(\text{(3 marks)}\)

(c) Using the phase portrait or otherwise, find the two equilibrium densities $M_+$ and $N_+$. \(\text{(2 marks)}\)

(iii) Calculate the Jacobian matrix, $J$, at a general equilibrium $(M_e, N_e)$. Substitute in the equilibria for $(M_+, 0)$ and $(0, N_+)$ and determine whether it is ever possible for one species to be driven to extinction. \(\text{(7 marks)}\)

Without calculating the equilibrium densities $M^*$ and $N^*$, use the Jacobian to show that $(M^*, N^*)$ is a stable node provided $w < vb/c$. How does this relate this to the condition found in part (ii)(b)? What can you conclude? \(\text{(6 marks)}\)
Consider a human population that is exposed to an infectious disease. A proportion \( v \in [0, 1] \) of newborn offspring are vaccinated at birth and are thus born immune. Partitioning the population in to either susceptible \((S)\), infected \((I)\) or recovered/immune \((R)\) compartments, the dynamics of this population are given by the ordinary differential equations,

\[
\frac{dS}{dt} = \mu (1 - v) - \beta SI - \mu S \quad (3)
\]

\[
\frac{dI}{dt} = \beta SI - (\gamma + \mu) I \quad (4)
\]

\[
\frac{dR}{dt} = \mu v + \gamma I - \mu R, \quad (5)
\]

where \( \mu, \beta, \gamma \) are positive constants, and the total population \( S + I + R = 1 \).

(i) Give biological definitions of the parameters \( \beta \) and \( \gamma \). \( (2 \text{ marks}) \)

(ii) Explain why this system may be fully described using only the equations for \( dS/dt \) and \( dI/dt \). \( (2 \text{ marks}) \)

(iii) Find the densities \((S_{df}, I_{df})\) at the disease-free equilibrium, and show that the endemic equilibrium is given by,

\[
S^* = \frac{\gamma + \mu}{\beta}, \quad I^* = \frac{\mu}{\beta} \left[ \frac{1 - v}{S^*} - 1 \right]. \quad (6)
\]

(5 marks)

(iv) (a) Find the Jacobian matrix of the system at a general equilibrium \((S_e, I_e)\). \( (2 \text{ marks}) \)

(b) Substitute both the disease-free, \((S_{df}, I_{df})\), and endemic, \((S^*, I^*)\), equilibria in to the Jacobian, and thus conclude whether the disease persists or dies out when: (1) \( v < (1 - S^*) \), and (2) \( v > (1 - S^*) \). \( (9 \text{ marks}) \)

(v) Let \( \mu = 0.1, \gamma = 0.9 \) and \( \beta = 2 \). Sketch a bifurcation diagram for this system, plotting both \( S_{df} \) and \( S^* \) as the parameter \( v \) is varied between 0 and 1 on the \( x \)-axis. Use a solid line to denote a stable equilibrium and a dashed line for an unstable equilibrium. Name the type of bifurcation that occurs at \( v = 0.5 \). \( (5 \text{ marks}) \)
The regulated transcription of a gene is represented by the differential equation
\[ \frac{dm}{dt} = -\mu m + f(t), \quad t \geq 0, \]
where \( m(t) \) represents the concentration of the mRNA transcript associated with the gene, and \( \mu \) is a positive constant.

(i) What are the meanings of the parameter \( \mu \) and the function \( f(t) \)?

(ii) Show that if \( m(0) = 0 \) and \( f(t) = \frac{1}{2} (1 + \cos \omega t) \), then
\[ m(t) = A + B (\mu \cos \omega t + \omega \sin \omega t) - Ce^{-\mu t}, \]
where \( A, B \) and \( C \) are constants that you should determine.

(iii) Show that, as \( t \to \infty \), \( m(t) \) approaches the periodic solution
\[ m_P(t) = A + \tilde{B} \cos [\omega (t - \tau)], \]
where \( \tilde{B} \) and \( \tau \) are constants that you should determine. Show that
\[ 0 < \tau < \frac{\pi}{2\omega}. \]
You may find the identity \( \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \) helpful.

(iv) Show that, as \( t \to \infty \), the ratio \( \rho \) between the maximum and minimum concentrations of mRNA is given by
\[ \rho = \frac{\sqrt{\mu^2 + \omega^2} + \mu}{\sqrt{\mu^2 + \omega^2} - \mu}. \]

(v) If the period of \( f(t) \) is 120 min, and \( \mu = 0.03 \text{ min}^{-1} \), show that, as \( t \to \infty \), peaks of mRNA concentration occur approximately 20 min after peaks of \( f(t) \), and that the fold-difference between maxima and minima of mRNA concentration is approximately 3.
A model for the expression of an autoregulatory gene is given by

\[
\frac{dM}{dt} = f(P) - \mu M \tag{7}
\]
\[
\frac{dP}{dt} = kM - \nu P, \tag{8}
\]

where \( M \) and \( P \) represent the concentration of mRNA and protein, and \( \mu, \nu \) and \( k \) are positive constants, and

\[
f(P) = \frac{\theta^m}{\theta^m + P^m}, \quad \theta > 0, \ m \geq 1.
\]

(i) Sketch \( f(P) \) for \( m = 1 \) and for \( m > 1 \). State whether the model represents autoactivation or autorepression. \( 5 \) marks

(ii) Sketch the nullclines for the equations (7) and (8) and show that the model has a unique steady state \((M^*, P_*)\). Show graphically that \( P_* \) is an increasing function of \( \theta \). Show that \( P_* \) is the positive solution of the equation

\[
P^{m+1} + \theta^m P = \frac{k}{\mu \nu} \theta^m. \tag{9}
\]

(6 marks)

(iii) By linearising the model around the steady state, show that the steady state is always stable, and that it is a stable spiral if

\[
-4k\phi > (\mu - \nu)^2,
\]

where \( \phi = \frac{df}{dP}(P_*) \). \( 7 \) marks

(iv) Show that \( \frac{df}{dP} = -m\theta^{-m}P^{m-1}f(P)^2 \).

By evaluating this expression at the steady state \( P_* \) and using the steady state conditions and (9), show that

\[
\phi = -m \left( \frac{\mu \nu}{k} \right)^2 \left( \frac{k}{\mu \nu} - P_* \right).
\]

Using your result from (ii), show that if \( \mu \neq \nu \), then the steady state is a stable spiral only for sufficiently small values of \( \theta \). \( 7 \) marks

End of Question Paper