Data provided: Formula sheet

MAS380

The University Of Sheffield.

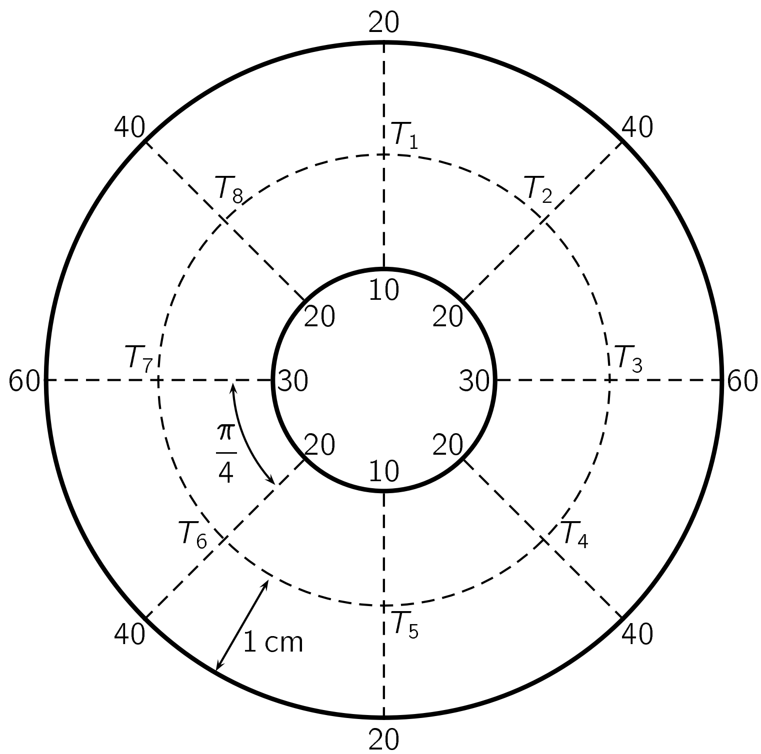
SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2017–18

Computational Engineering Mathematics Three hours

Marks will be awarded for your best FOUR answers.
The maximum possible mark for the paper is 100.

1 The figure shows an annular plate (bounded by concentric circles of radius 1 cm and radius 3 cm) made of a homogeneous isotropic material. In polar coordinates (r, θ) the plate is divided into intervals with spacing of $\Delta r = 1$ cm in the radial (r) direction, and $\Delta\theta = \pi/4$ radians in the θ direction. The temperature $T(r, \theta)$ in this plate satisfies the indicated boundary conditions (given in °C) and has reached a steady-state condition so that it is described by Laplace's equation, which in polar coordinates is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0.$$



(a) Draw a sketch of the solution domain, showing clearly the lines of symmetry for $T(r, \theta)$, and indicating which of the unknown temperatures $T_1, T_2, \ldots, T_8$ are equal to each other. ***(5 marks)***

 Question 1 continued on next page

(b) If $\Delta\theta$ is small and $T_{i,j} = T(r_i, \theta_j)$, give an argument for why

$$\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}(r_i, \theta_j) \approx \frac{1}{r_i^2}\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta\theta)^2}.$$

(2 marks)

(c) Using the approximate result in part (b), and a central difference for $\dfrac{\partial T}{\partial r}$, find the finite difference equations required to find estimates of the nodal temperatures T_1, T_2 and T_3. Show that these can be expressed in the form $A\mathbf{T} = \mathbf{b}$, where

$$A = \begin{bmatrix} 16 + 4\pi^2 & -16 & 0 \\ -8 & 16 + 4\pi^2 & -8 \\ 0 & -16 & 16 + 4\pi^2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 65\pi^2 \\ 130\pi^2 \\ 195\pi^2 \end{bmatrix},$$

with units in °C. ***(10 marks)***

Hence, using Gaussian elimination or otherwise and taking π^2 to be given by 10, find the values of T_1, T_2 and T_3 which follow *exactly* from these equations. ***(8 marks)***

2 The temperature $T(x, t)$ satisfies the heat conduction equation

$$\frac{\partial T}{\partial t} = 2\frac{\partial^2 T}{\partial x^2} \qquad (0 \leqslant x \leqslant 1, \quad t > 0). \tag{1}$$

(a) If $T_{i,j} = T(x_i, t_j)$, with $i = 0$ and $i = N$ corresponding to $x = 0$ and $x = 1$, respectively, and $j = 0$ corresponding to $t = 0$, use backward differences for time derivatives and central differences for space derivatives to derive the implicit scheme

$$-2k\, T_{i+1,j} + (1 + 4k)\, T_{i,j} - 2k\, T_{i-1,j} = T_{i,j-1}$$

for $i = 1, \ldots, N-1$ and $j = 1, 2, \ldots,$ where

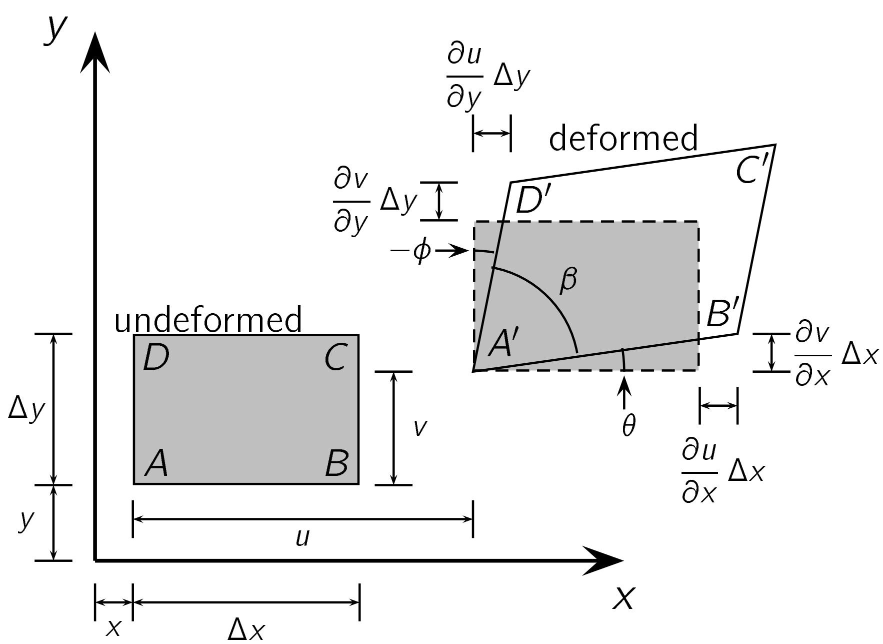
$$k = \frac{\Delta t}{(\Delta x)^2}.$$ ***(4 marks)***

(b) Equation (1) is to be solved over $0 \leqslant x \leqslant 1$, with boundary conditions $T(0, t) = 20$ and $T(1, t) = 10$, and initial temperature distribution $T(x, 0) = 20 - 10x^2$. Taking $\Delta x = 0.2$ and $\Delta t = 0.1$, use the implicit scheme in part (a) to write down the system of equations for the temperature at $x = 0.2, 0.4, 0.6, 0.8$ and time $t = 0.1$. (Note that you *do not* need to solve the equations.) ***(11 marks)***

(c) Show from your answer to part (b) that the Jacobi iteration equations to find the $(n+1)$th iteration from the nth iteration are

$$\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{bmatrix}^{(n+1)} = \frac{1}{11}\begin{bmatrix} 119.6 \\ 18.4 \\ 16.4 \\ 63.6 \end{bmatrix} + \frac{5}{11}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{3,1} \\ T_{4,1} \end{bmatrix}^{(n)}$$ ***(10 marks)***

3 (a) Referring to the figure, for small displacements u and v, define the normal strain ε_{yy} and the engineering shear strain γ_{xy}. ***(3 marks)***



Hence show that

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \text{and} \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

where u is the displacement of the body $ABCD$ in the x-direction, and v is the displacement in the y-direction. ***(12 marks)***

(b) The matrix relating the *engineering* strains to the stresses for an isotropic material is given by

$$\begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

If $E = 31.0\,\text{GPa}$, $\nu = 0.157$, $\varepsilon_{xx} = 927 \times 10^{-6}$, $\varepsilon_{yy} = 0.31\varepsilon_{xx}$, $\varepsilon_{zz} = -0.23\varepsilon_{xx}$, $\varepsilon_{xy} = 327 \times 10^{-6}$, $\varepsilon_{yz} = -453 \times 10^{-6}$ and $\varepsilon_{zx} = 53 \times 10^{-6}$ at a particular point in the material, calculate the stress at that point, giving your answers to 3 significant figures. ***(10 marks)***

4 An incompressible fluid of constant density ρ is contained between vertical plane solid boundaries at $x = -a$ and $x = a$, where a is a positive constant. The boundaries both move vertically upwards (in the z-direction) with constant speed U.

The pressure gradient in the vertical direction is zero, i.e.

$$\frac{\partial p}{\partial z} = 0,$$

and the fluid velocity $\mathbf{v}$ is given by

$$\mathbf{v} = w(x)\,\mathbf{k},$$

where $\mathbf{k}$ is the unit vector in the z-direction.

The velocity $\mathbf{v}$ satisfies

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{v} - \rho g\mathbf{k},$$

where μ is the constant viscosity, g is the constant gravitational acceleration, and

$$w = U \quad \text{when} \quad x = \pm a.$$

(a) Show that

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{0},$$

and hence that

$$\mathbf{0} = -\nabla p + \mu\frac{d^2 w}{dx^2}\mathbf{k} - \rho g\mathbf{k}.$$ ***(5 marks)***

Find $w(x)$. ***(10 marks)***

(b) Find the fluid flux $\int_{-a}^{a} w\,dx$ across a horizontal plane, per unit distance in the y-direction. ***(5 marks)***

Show that if the flux is zero then

$$U = \frac{\rho g a^2}{3\mu}.$$ ***(2 marks)***

Using this, find the fluid velocity midway between the boundaries. ***(3 marks)***

5 (a) A rectangular surface on a positive x-plane is defined by $x = 0$, $0 \leqslant y \leqslant 1$ and $0 \leqslant z \leqslant 2$ (where all lengths are measured in m). The stress tensor depends on y and is given by (in units of Pa)

$$[\sigma] = \begin{bmatrix} 6y^2 & 2y & 4y^3 \\ 2y & 1 & -3y^2 \\ 4y^3 & -3y^2 & 1 \end{bmatrix}.$$

Show that the total stress force $\mathbf{f}$ on the rectangular surface is

$$\mathbf{f} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{N}.$$ ***(6 marks)***

(b) In index notation the i-component of the curl of a vector field $\mathbf{F}$ may be expressed as

$$(\nabla \times \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j},$$

where ε_{ijk} is the Levi-Civita tensor.

Show, using index notation, that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$ ***(10 marks)***

(You may use $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, where δ_{ij} is the Kronecker delta tensor.)

(c) (i) Using index notation, show that for any vector fields $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$$

and

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$ ***(6 marks)***

(ii) If V is a closed volume bounded by a surface S, Gauss' theorem states that

$$\int_V \nabla \cdot \mathbf{F} \, dV = \int_S \mathbf{F}.\hat{\mathbf{n}} \, dS$$

for a vector field $\mathbf{F}$, where $\hat{\mathbf{n}}$ is the unit normal to S, pointing out of V.

By choosing

$$\mathbf{F} = \mathbf{a} \times \mathbf{u},$$

where $\mathbf{a}$ is an arbitrary constant vector, and using the results of part (i), show that

 Question 5 continued on next page

5 (continued)

$$\int_V \nabla \times \mathbf{u} \, dV = \int_S \hat{\mathbf{n}} \times \mathbf{u} \, dS. \qquad \textbf{\textit{(3 marks)}}$$

End of Question Paper

Formula Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}$$

Forward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i,j}}{\Delta x}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}$$

Backward difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i,j} - U_{i-1,j}}{\Delta x}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x}(x_i,\ t_j) \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2}(x_i,\ t_j) \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{(\Delta x)^2}$$

Relation between different parameters:

A number of relationships between E, ν, K, λ and μ hold and are summarized in Table 1. μ ($\equiv G$) is the elastic shear modulus, K the elastic bulk modulus, E the elastic stiffness (or Young's Modulus) and ν Poisson's ratio.

	E	ν	K	λ	$\mu \equiv G$
E, ν	-	-	$\frac{E}{3(1-2\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$
E, K	-	$\frac{3K-E}{6K}$	-	$\frac{K(9K-3E)}{9K-E}$	$\frac{3KE}{9K-E}$
K, μ	$\frac{9\mu K}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	-	$K - \frac{2\mu}{3}$	-

Table 1: The relations between the properties of elastic bodies.