Data provided: Formula sheet

# MAS381

The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2017–18**

**Mathematics III (Electrical)** **2 hours**

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** (i) Use the residue theorem to find the value of the integral

$$I = \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$$

***(13 marks)***

(ii) Let us consider the function $u(x, y) = x^2 - y^2 + 5x - 6y$. Show that $u(x, y)$ is a harmonic function and find a harmonic conjugate of $u(x, y)$. Determine any constant involved in the solution taking into account that the function $f(x, y) = u(x, y) + iv(x, y)$ takes the value $-1 + 3i$ when $(x, y) = (1, 1)$.

***(12 marks)***

**2** (i) Let us consider the vector field

$$\mathbf{F}(x, y, z) = \left( \frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right)$$

(a) Calculate $\nabla \cdot \mathbf{F}$ and show that $\mathbf{F}(x, y, z)$ is an irrotational vector field. ***(4 marks)***

(b) Find the scalar potential function $\phi(x, y, z)$ so that $\mathbf{F} = \nabla(\phi)$. ***(10 marks)***

(ii) Find all the poles of the function

$$f(z) = \frac{z^2}{z^4 + 2z^2 - 3},$$

and plot them on an Argand diagram. Hence evaluate the integral $\oint_C f(z)dz$, writing your solutions in the form $a + ib$, where $a$ and $b$ are real and

(a) $C$ is the circle $|z| = \sqrt{2}$,

(b) $C$ is the circle $|z + 3i/2| = 3/2$. ***(11 marks)***

**Question 2 continued on next page**

**2** (continued)

(N.B. All contour integrals should be evaluated in the *counterclockwise* direction)

**3** (i) Expand the function

$$f(z) = \frac{z-1}{3z^2 i - 5z - 2i}$$

into Laurent series in the interval $2/3 \leq |z| \leq 1$. ***(17 marks)***

(ii) Determine how the bilinear mapping

$$w = \frac{z}{1-z}$$

is mapping the unit disk $|z| \leq 1$ in the $w$ plane. Sketch your results in the $z$ and $w$ plane. ***(8 marks)***

**4** (i) A charged particle is travelling subject to an electric force field $\mathbf{E} = (x, y, -xy)$. Calculate the work done by this force on the particle when it moves

(a) Along the line segment from $(-1, 2, 0)$ to $(3, 0, 1)$

(b) Along the arc of a circle of unit radius centered on $(1, 0, 0)$, between the points corresponding to $(1, 1, 0)$ and $(0, 0, 0)$ in the counterclockwise direction. ***(19 marks)***

(ii) Prove that

$$\Re\left\{\frac{z}{1-z}\right\} > -\frac{1}{2}$$

for any complex number $z$ that satisfies the condition $|z| < 1$ (here $\Re$ denotes the real part of a complex quantity). ***(6 marks)***

## End of Question Paper

## Formula sheet

- The general formula for the residue at a pole $z_0$, of order $m$ is

$$\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\right\}$$

- Useful identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}; \quad \sin m\theta\cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]$$

- The polar and spherical area elements are given by

$$dA = r dr d\theta, \quad dA = r^2 \sin\phi \, d\phi \, d\theta$$