The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2017–18**

**Topics in Advanced Fluid Mechanics** **2 hours 30 minutes**

*Marks will be awarded for your best **four** answers.*

**1** For an incompressible flow $\nabla \cdot \boldsymbol{u} = 0$, we consider the vorticity equations

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u},$$

and the equations for the impulse

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla \boldsymbol{u})^T \boldsymbol{\gamma} + \nabla \lambda,$$

where $T$ denotes matrix transpose and $\lambda = \lambda(\boldsymbol{x}, t)$ an arbitrary function.

(i) Show that

$$\frac{D\boldsymbol{\gamma} \cdot \boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\lambda$$

holds. ***(5 marks)***

(ii) Deduce

$$\frac{d}{dt}\int_{\mathbb{R}^3} \boldsymbol{\gamma} \cdot \boldsymbol{\omega} d\boldsymbol{x} = 0.$$

***(10 marks)***

(iii) By the Cauchy formula, show that the following identity

$$\boldsymbol{\omega}(\boldsymbol{x}, t) \cdot \nabla = \boldsymbol{\omega}_0(\boldsymbol{a}) \cdot \frac{\partial}{\partial \boldsymbol{a}}$$

holds, where $\boldsymbol{\omega}_0$ denotes the initial value of the vorticity. On this basis, show that the 'helicity' $\boldsymbol{\gamma} \cdot \boldsymbol{\omega}$ behaves as a linear function in time, if we choose

$$\frac{D\lambda}{Dt} = 0.$$

***(10 marks)***

**2** Consider the Burgers equation with an extra term.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + a\frac{\partial}{\partial x}(xu), \tag{1}$$

where $a(\geq 0)$ is a constant. We consider a transformation

$$u = -2\nu \frac{\partial}{\partial x} \log \psi$$

for (1).

(i) For $a = 0$ (the case of the original Burgers equation), show that the following identity holds

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = -2\nu \frac{\partial}{\partial x}\left(\frac{\frac{\partial \psi}{\partial t} - \nu \frac{\partial^2 \psi}{\partial x^2}}{\psi}\right).$$

***(10 marks)***

(ii) For $a > 0$, by including the additional term, show that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} - a\frac{\partial}{\partial x}(xu) = -2\nu \frac{\partial}{\partial x}\left(\frac{\frac{\partial \psi}{\partial t} - \nu \frac{\partial^2 \psi}{\partial x^2} - ax\frac{\partial \psi}{\partial x}}{\psi}\right)$$

holds.

***(5 marks)***

(iii) Show that a steady solution of

$$\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2} + ax\frac{\partial \psi}{\partial x} \quad (a > 0)$$

is given by

$$\psi(x) = c_1 \int_{-\infty}^{x} e^{-\frac{ay^2}{2\nu}}\, dy + c_2,$$

where $c_1$ and $c_2$ are constants. ***(5 marks)***

(iv) Give a rough sketch of a graph of the variable $\psi$ associated with the steady solution and state what it represents physically when $a/\nu$ is much larger than 1. ***(5 marks)***

**3** Consider a model equation for vorticity $\omega(x,t)$

$$\frac{\partial \omega}{\partial t} = \omega H[\omega],$$

with an initial condition $\omega(x,0) = \sin(x)$ and the Hilbert transform

$$H[\omega](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy,$$

defined by a principal-value integral. We decompose $\omega(x,t) = \omega_+(x,t) + \omega_-(x,t)$, where

$$\omega_+(x,t) = \sum_{n=1}^{\infty} A_n(t) e^{inx}, \ \omega_-(x,t) = \sum_{n=1}^{\infty} A_{-n}(t) e^{-inx}$$

denote the upper- and lower- analytic components, respectively. Here $A_0(t) = 0$ for all $t$, $A_1(0) = \dfrac{1}{2i}$ and $A_n(t) = A^*_{-n}(t)$ for $n \geq 1$, where * denotes complex conjugate.

You are given that they satisfy

$$\frac{\partial \omega_+}{\partial t} = -i\omega_+^2, \quad \frac{\partial \omega_-}{\partial t} = i\omega_-^2.$$

(i) Derive the following equation for $A_n$

$$\frac{dA_n}{dt} = -i \sum_{p=1}^{n-1} A_p A_{n-p}, \ (n \geq 1).$$

*(7 marks)*

(ii) By direct compuations, show that

$$A_n(t) = \frac{(-t)^{n-1}}{i2^n}, \ (n \geq 1)$$

is a solution. *(7 marks)*

(iii) By using $\omega_-(x,t) = (\omega_+(x,t))^*$, show that

$$\omega(x,t) = \sum_{n=0}^{\infty} \left(\frac{-t}{2}\right)^n \sin(n+1)x.$$

*(7 marks)*

(iv) Find a corresponding representation for $H[\omega]$, which is a Taylor series in $t$ and a trigonometric series in $x$. *(4 marks)*

**4** We consider the motion of an elliptic vortex of uniform density $\omega$ governed by

$$\begin{cases} \dfrac{da}{dt} = ea\cos 2\theta, \quad \dfrac{db}{dt} = -eb\cos 2\theta, \\ \dfrac{d\theta}{dt} = -e\dfrac{a^2+b^2}{a^2-b^2}\sin 2\theta + \dfrac{ab\omega}{(a+b)^2} + \gamma. \end{cases}$$

Here $a(t), b(t)$ and $\theta(t)$ denote the major axis, the minor axis and the angle the major axis makes with the $x$-axis. The parameters $e$ and $\gamma$ denote an externally imposed strain rate and vorticity, respectively.

(i) Show that the product $a(t)b(t)$ is constant and state its meaning.

***(3 marks)***

(ii) Derive a set of equations for $r(t) = a(t)/b(t)$ and $\theta(t)$ as follows

$$\begin{cases} \dfrac{dr}{dt} = 2er\cos 2\theta, \\ \dfrac{d\theta}{dt} = -e\dfrac{r^2+1}{r^2-1}\sin 2\theta + \dfrac{\omega r}{(r+1)^2} + \gamma. \end{cases}$$

***(4 marks)***

(iii) By evaluating $\dfrac{d\theta}{dr} = \dfrac{d\theta}{dt} \Big/ \dfrac{dr}{dt}$, derive

$$\frac{dG}{dr} + G\frac{d\log F}{dr} = \frac{\omega}{e}\frac{1}{(r+1)^2} + \frac{\gamma}{er},$$

where $F(r) \equiv \dfrac{r^2-1}{r}, G(\theta) \equiv \sin 2\theta.$ ***(5 marks)***

(iv) Derive a first integral

$$FG = \frac{\omega}{e}\log\frac{(r+1)^2}{r^2} + \frac{\gamma}{e}\left(r + \frac{1}{r}\right) + C,$$

where $C$ is an integration constant. ***(5 marks)***

(v) Derive

$$\sin 2\theta = \frac{\omega}{e}\underbrace{\left(\frac{r}{r^2-1}\log\frac{(r+1)^2}{4sr} + \frac{\gamma}{\omega}\frac{r-1}{r+1}\right)}_{\equiv g(r)},$$

by suitably choosing a new constant $s$. ***(4 marks)***

(vi) For $g(r)$ defined in (v), show that

$$\frac{dg}{dr} = -\frac{r^2+1}{(r^2-1)^2}\log\frac{(r+1)^2}{4sr} + \frac{1}{(r+1)^2} + \frac{2\gamma}{\omega}\frac{1}{(r+1)^2}$$

and hence deduce

$$\frac{d\theta}{dt} = \omega r\frac{dg}{dr}.$$

***(4 marks)***

**5** The motion of a vortex layer of uniform strength is governed by

$$\frac{\partial z(\alpha,t)^*}{\partial t} = -\frac{\mathrm{i}}{2\pi} ⨍_{-\infty}^{\infty} \frac{d\beta}{z(\alpha,t) - z(\beta,t)}, \tag{1}$$

where

$$z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$$

denotes the position of a fluid particle $\alpha$ on the layer, $*$ complex conjugate and $⨍_{-\infty}^{\infty} = \lim_{L\to\infty} ⨍_{-L}^{L}$ a principal-value integral. We study the stability property of the vorticity layer around a flat state $z_0(\alpha) = \alpha$.

(i) By setting $z(\alpha) = \alpha + if(\alpha,t)$, where $f(\alpha,0) = 0$, derive from (1)

$$\frac{\partial f(\alpha)^*}{\partial t} = \frac{1}{2\pi} ⨍_{-\infty}^{\infty} \frac{d\beta}{\alpha - \beta + i(f(\alpha) - f(\beta))}. \tag{2}$$

***(3 marks)***

(ii) Derive a linearised equation from (2)

$$\frac{\partial f(\alpha)^*}{\partial t} = -\frac{i}{2\pi} ⨍_{-\infty}^{\infty} \frac{f(\alpha) - f(\beta)}{(\alpha - \beta)^2} d\beta. \tag{3}$$

***(6 marks)***

(iii) Assuming boundedness $f(\alpha) < M$ with a constant $M$, use integration by parts to rewrite (3) as

$$\frac{\partial f^*}{\partial t} = -\frac{i}{2} H\left[\frac{\partial f}{\partial \alpha}\right], \tag{4}$$

where $H$ denotes the Hilbert transform (see **Question** 3 for definition).
***(5 marks)***

(iv) Derive from (4)

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4}\frac{\partial^2 f}{\partial \alpha^2}.$$

***(6 marks)***

(v) Consider a Fourier mode of the form

$$f = \exp(ik\alpha + \lambda t),$$

where $k$ is the wavenumber and $\lambda$ the growth rate. Express $\lambda$ as a function of $k$ and decide whether it is stable or unstable. ***(5 marks)***

## End of Question Paper