Data provided: Formula Sheet

MAS420

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

MAS420 Signal Processing **2 hours**

Attempt ***ALL*** *questions.*

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Given that $f(t) \longleftrightarrow F(\omega)$ show that $f^*(-t) \longleftrightarrow F^*(\omega)$. ***(2 marks)***

(ii) Given that $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$, prove that

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega.$$

Hence show that if $f(t)$ has energy E, then

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega.$$

(4 marks)

(iii) Find the Fourier transform, $F(\omega)$, of

$$f(t) = \operatorname{sinc}(t)\sin(ct).$$

Sketch $F(\omega)$ for (a) $c \geq 1$ and (b) $c < 1$. Show that the energy of this signal is independent of c if $c \geq 1$, but is proportional to c for $0 \leq c < 1$.

(8 marks)

Find also the energy of the signal

$$f(t) = \operatorname{sinc}(t)\cos(ct).$$

(5 marks)

Hence show that the energy E of the signal

$$f(t) = \operatorname{sinc}(t)\sin(ct + \phi),$$

where ϕ is a constant, is given by

$$E = \begin{cases} \dfrac{\pi}{2} & c \geq 1 \\[2ex] \dfrac{\pi}{2}(1 + (c-1)\cos(2\phi)) & c < 1. \end{cases}$$

(6 marks)

2 (i) Use the signal $f(t) = e^{i\omega t}$ as input to a linear shift-invariant system to define the system transfer function (STF). ***(2 marks)***

(ii) Prove that if $g(t) = f * h(t)$, then $G(\omega) = F(\omega)H(\omega)$, where signals are indicated by lower case letters and their Fourier transforms by upper case. ***(3 marks)***

(iii) Given a periodic function $f_T(t)$ of period T, show that its Fourier coefficients, c_n, are given by

$$c_n = \frac{1}{T}F(n\sigma)$$

where $F(\omega)$ is the Fourier transform of $f(t) = p_{\frac{T}{2}}(t)f_T(t)$, and $\sigma = \dfrac{2\pi}{T}$. ***(3 marks)***

 Question 2 continued on next page

2 (continued)

(iv) A system operates as shown in Figure 1.

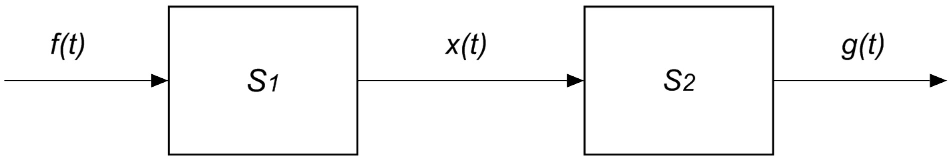

Figure 1

S_1 is a subsystem which has as output

$$x(t) = \frac{f(t+\tau) - 2f(t) + f(t-\tau)}{\tau^2}$$

where τ is a constant. S_2 is a subsystem which is an ideal low-pass filter with cut-off frequency Ω rad s^{-1}, i.e. $H_2(\omega) = p_\Omega(\omega)$.

(a) Show that the STF of the system is

$$H(\omega) = \frac{-4}{\tau^2} \sin^2 \frac{\omega\tau}{2} p_\Omega(\omega)$$

(5 marks)

(b) Using frequency domain methods, find the output $g(t)$ if the input, $f(t)$, is the periodic signal shown in Figure 2; $\tau = 1$ and $\Omega = \frac{5\pi}{4}$. (Hint: first find the Fourier series of $f(t)$). Show that $g(t)$ can be written in the form

$$g(t) = -\frac{8}{\pi^2}\left[\cos\frac{\pi t}{2} + \cos \pi t\right].$$

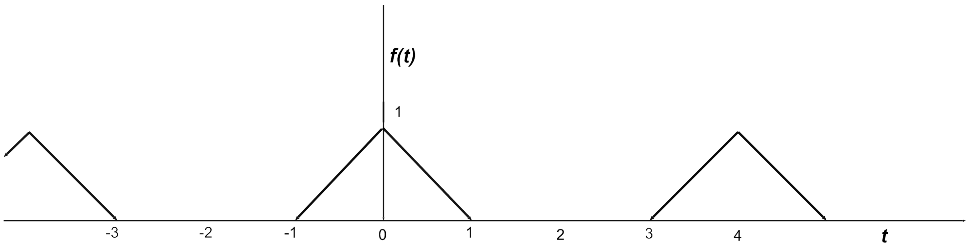

Figure 2

(9 marks)

(c) Find the form of the STF as $\tau \to 0$ and describe what this limiting signal does to the input signals (thinking about the time-domain definition of the system may help you). ***(3 marks)***

3 (i) Find the Fourier transform of the signal $e^{-a|t|}$, where $a > 0$. ***(4 marks)***

(ii) Prove that if $f(t) \leftrightarrow F(\omega)$, $F(\omega)$ is real and $F(\omega) \geq 0$ for all ω, then $f(0) \geq |f(t)|$ for all t. ***(2 marks)***

(iii) Define the equivalent rectangle resolution τ of a signal, making clear the conditions under which it is defined. ***(3 marks)***

The time-bandwidth theorem states that if $f(t)$ is an Ω-BL function for which the equivalent rectangle resolution is defined, then the time-bandwidth product, $\tau\Omega$ satisfies $\tau\Omega \geq \pi$ (equivalently $\tau B \geq \frac{1}{2}$ if B is the bandwidth in Hz), and $\tau\Omega = \pi$ if and only if $f(t) = k\,\text{sinc}(\Omega t)$ where k is a constant.

The signal $f(t) = \dfrac{a}{\pi(a^2+t^2)}$ is input to a linear system with system transfer function $H(\omega) = p_b(\omega)$ where $a > 0, b > 0$. Show that the output $g(t)$ meets the conditions of the time bandwidth theorem and find its equivalent rectangle resolution. Show also that as $a \to 0$ the time-bandwidth product tends to π and explain why this happens. ***(16 marks)***

4 (i) Show that, if $f(t) \leftrightarrow F(\omega)$ and

$$F_s(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} F(\omega - m\sigma),$$

where $\sigma T = 2\pi$, then

$$F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT) e^{-inT\omega}$$

(the Poisson sum formula). ***(6 marks)***

Use this to show that, if

$$g(t) = \sum_{n=-\infty}^{\infty} f(nT) \operatorname{sinc} \frac{\sigma}{2}(t - nT)$$

(the sinc interpolation formula), then $G(\omega) = p_{\sigma/2}(\omega) \sum_{n=-\infty}^{\infty} F(\omega - n\sigma)$.

(3 marks)

Using this result, with the aid of a clear diagram, state the conditions under which $g(t) = f(t)$. ***(3 marks)***

(ii) Find the Fourier transform of the signal $f(t) = \operatorname{sinc} 2t + \operatorname{sinc} 4t$ and hence find its Nyquist frequency in Hz (a clear diagram will help).

(5 marks)

This signal is sampled with sample spacing $T = \frac{\pi}{3}$, and the samples are used to form a signal $g(t)$ using the sinc interpolation formula. Making use of clear comprehensive diagrams, show that

$$g(t) = \frac{3}{2} \operatorname{sinc} 3t + \frac{1}{2} \operatorname{sinc} 2t.$$

(8 marks)

End of Question Paper

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a} p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a} q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a + i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t - t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t - t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega - \omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$

`[fsheet.tex: 24th August 2011, Rev. 1.1]`