Given that \( f(t) \leftrightarrow F(\omega) \) show that \( f^*(-t) \leftrightarrow F^*(\omega) \).

(ii) Given that \( f(t) \leftrightarrow F(\omega) \) and \( g(t) \leftrightarrow G(\omega) \), prove that

\[
\int_{-\infty}^{\infty} f(t)g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G^*(\omega) \, d\omega.
\]

Hence show that if \( f(t) \) has energy \( E \), then

\[
E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega.
\]

(iii) Find the Fourier transform, \( F(\omega) \), of

\[ f(t) = \text{sinc}(t) \sin(ct). \]

Sketch \( F(\omega) \) for (a) \( c \geq 1 \) and (b) \( c < 1 \). Show that the energy of this signal is independent of \( c \) if \( c \geq 1 \), but is proportional to \( c \) for \( 0 \leq c < 1 \).

Find also the energy of the signal

\[ f(t) = \text{sinc}(t) \cos(ct). \]

Hence show that the energy \( E \) of the signal

\[ f(t) = \text{sinc}(t) \sin(ct + \phi), \]

where \( \phi \) is a constant, is given by

\[
E = \begin{cases} 
\frac{\pi}{2} & c \geq 1 \\
\frac{\pi}{2}(1 + (c - 1)\cos(2\phi)) & c < 1.
\end{cases}
\]
2  
(i) Use the signal $f(t) = e^{j\omega t}$ as input to a linear shift-invariant system to define the system transfer function (STF). \hspace{2cm} (2 \text{ marks})

(ii) Prove that if $g(t) = f * h(t)$, then $G(\omega) = F(\omega)H(\omega)$, where signals are indicated by lower case letters and their Fourier transforms by upper case. \hspace{2cm} (3 \text{ marks})

(iii) Given a periodic function $f_T(t)$ of period $T$, show that its Fourier coefficients, $c_n$, are given by

$$c_n = \frac{1}{T} F(n\sigma)$$

where $F(\omega)$ is the Fourier transform of $f(t) = p_T(t)f_T(t)$, and $\sigma = \frac{2\pi}{T}$. \hspace{2cm} (3 \text{ marks})
(iv) A system operates as shown in Figure 1.

\[ f(t) \xrightarrow{S_1} x(t) \xrightarrow{S_2} g(t) \]

Figure 1

\( S_1 \) is a subsystem which has as output

\[ x(t) = \frac{f(t + \tau) - 2f(t) + f(t - \tau)}{\tau^2} \]

where \( \tau \) is a constant. \( S_2 \) is a subsystem which is an ideal low-pass filter with cut-off frequency \( \Omega \) rad s\(^{-1}\), i.e. \( H_2(\omega) = p_{\Omega}(\omega) \).

(a) Show that the STF of the system is

\[ H(\omega) = -\frac{4}{\tau^2} \sin^2 \frac{\omega \tau}{2} p_{\Omega}(\omega) \]

(5 marks)

(b) Using frequency domain methods, find the output \( g(t) \) if the input, \( f(t) \), is the periodic signal shown in Figure 2; \( \tau = 1 \) and \( \Omega = \frac{5\pi}{4} \).

(Hint: first find the Fourier series of \( f(t) \)). Show that \( g(t) \) can be written in the form

\[ g(t) = -\frac{8}{\pi^2} \left[ \cos \frac{\pi t}{2} + \cos \pi t \right]. \]

(9 marks)

(c) Find the form of the STF as \( \tau \to 0 \) and describe what this limiting signal does to the input signals (thinking about the time-domain definition of the system may help you).

(3 marks)
3. (i) Find the Fourier transform of the signal $e^{-a|t|}$, where $a > 0$. (4 marks)

(ii) Prove that if $f(t) \leftrightarrow F(\omega)$, $F(\omega)$ is real and $F(\omega) \geq 0$ for all $\omega$, then $f(0) \geq |f(t)|$ for all $t$. (2 marks)

(iii) Define the equivalent rectangle resolution $\tau$ of a signal, making clear the conditions under which it is defined. (3 marks)

The time-bandwidth theorem states that if $f(t)$ is an $\Omega$-BL function for which the equivalent rectangle resolution is defined, then the time-bandwidth product, $\tau \Omega$ satisfies $\tau \Omega \geq \pi$ (equivalently $\tau B \geq \frac{1}{2}$ if $B$ is the bandwidth in Hz), and $\tau \Omega = \pi$ if and only if $f(t) = k \text{sinc}(\Omega t)$ where $k$ is a constant.

The signal $f(t) = \frac{a}{\pi(a^2 + t^2)}$ is input to a linear system with system transfer function $H(\omega) = p_b(\omega)$ where $a > 0, b > 0$. Show that the output $g(t)$ meets the conditions of the time bandwidth theorem and find its equivalent rectangle resolution. Show also that as $a \to 0$ the time-bandwidth product tends to $\pi$ and explain why this happens. (16 marks)
(i) Show that, if \( f(t) \leftrightarrow F(\omega) \) and
\[
F_s(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} F(\omega - m\sigma),
\]
where \( \sigma T = 2\pi \), then
\[
F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT)e^{-inT\omega}
\]
(the Poisson sum formula). \( \text{(6 marks)} \)

Use this to show that, if
\[
g(t) = \sum_{n=-\infty}^{\infty} f(nT) \text{sinc} \frac{\sigma}{2}(t - nT)
\]
(the sinc interpolation formula), then \( G(\omega) = p_{\sigma/2}(\omega) \sum_{n=-\infty}^{\infty} F(\omega - n\sigma) \).
\( \text{(3 marks)} \)

Using this result, with the aid of a clear diagram, state the conditions under which \( g(t) = f(t) \).
\( \text{(3 marks)} \)

(ii) Find the Fourier transform of the signal \( f(t) = \text{sinc} 2t + \text{sinc} 4t \) and hence find its Nyquist frequency in Hz (a clear diagram will help).
\( \text{(5 marks)} \)

This signal is sampled with sample spacing \( T = \frac{\pi}{3} \), and the samples are used to form a signal \( g(t) \) using the sinc interpolation formula. Making use of clear comprehensive diagrams, show that
\[
g(t) = \frac{3}{2} \text{sinc} 3t + \frac{1}{2} \text{sinc} 2t.
\]
\( \text{(8 marks)} \)

End of Question Paper
Function Definitions:
Rectangular pulse:
\[
p_a(t) = \begin{cases} 
1 & |t| \leq a \\
0 & |t| > a 
\end{cases}
\]
Triangular pulse:
\[
q_a(t) = \begin{cases} 
1 - |t|/a & |t| \leq a \\
0 & |t| > a 
\end{cases}
\]
Step function:
\[
U(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0 
\end{cases}
\]
Fourier Transform Pairs:
\[
p_a(t) \leftrightarrow 2a \text{sinc}(a\omega) 
q_a(t) \leftrightarrow a \text{sinc}^2(a\omega/2) 
sinc(at) \leftrightarrow \frac{p_a(\omega)}{\pi t} 
sinc^2(at) \leftrightarrow \frac{q_{2a}(\omega)}{\pi t} 
e^{-at}U(t) \leftrightarrow \frac{1}{a + i\omega} 
\delta(t) \leftrightarrow 1 
\delta(t-t_0) \leftrightarrow e^{-i\omega t_0} 
1 \leftrightarrow 2\pi \delta(\omega) 
e^{i\omega t} \leftrightarrow 2\pi \delta(\omega - \omega_0) 
e^{-t^2/2\sigma^2} \leftrightarrow \sigma \sqrt{2\pi} e^{-\sigma^2 \omega^2/2}
\]
Fourier transform:
\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt
\]
Inverse Fourier transform:
\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega
\]
Duality theorem: If \( f(t) \leftrightarrow F(\omega) \) then \( F(t) \leftrightarrow 2\pi f(-\omega) \)
Scaling: If \( f(t) \leftrightarrow F(\omega) \) then \( f(at) \leftrightarrow \frac{1}{|a|} F(\omega/a) \).
Translation: If \( f(t) \leftrightarrow F(\omega) \) then \( f(t-t_0) \leftrightarrow e^{-i\omega t_0} F(\omega) \).
Frequency Shift: If \( f(t) \leftrightarrow F(\omega) \) then \( e^{i\omega t} f(t) \leftrightarrow F(\omega - \omega_0) \)
**Fourier Series:** If $f_T(t)$ is periodic with period $T$ then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\sigma t}$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-in\sigma t} \, dt$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos n\sigma t + b_n \sin n\sigma t \right)$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \cos n\sigma t \, dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \sin n\sigma t \, dt$$

**Parseval’s Theorem:** If $V$ is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for $V$ and $f = \sum c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

**Plancherel’s Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \, d\omega$$

**Energy Theorem:** If $f(t) \leftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

**Convolution Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) \, ds \leftrightarrow F(\omega)G(\omega)$$

**Product Theorem:** If $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$ then

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) * G(\omega).$$