The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017–18**

Analytic Number Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Note that the questions do not carry equal marks: Q1 is worth 34 marks, Q2 is worth 35 marks, Q3 is worth 21 marks, and Q4 is worth 10 marks.

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1 (i) Imitate Euclid's proof to show that there are infinitely many primes that are equal to $-1 \bmod 8$.
(Hint: You can use the fact that for any integer n, every odd prime divisor of $16n^2 - 2$ is equal to $\pm 1 \bmod 8$.) ***(6 marks)***

(ii) Let $f(x) \in \mathbb{Z}[x]$ be a non-constant polynomial. Let us say that a prime p *numerically divides* $f(x)$ if there is an integer n such that p divides $f(n)$. Let $S(f)$ denote the set of primes that numerically divide $f(x)$.

(a) Show that if $f(0) = 0$, then $S(f)$ is infinite. ***(2 marks)***

(b) Show that if $f(0) \neq 0$, then $S(f)$ is infinite.
(Hint: Prove by contradiction. Put $c = f(0)$. For a smart choice of N, write $f(Ncx) = cg(x)$ and inspect $g(x)$.) ***(8 marks)***

(iii) Show that there are arbitrarily long gaps between primes. ***(5 marks)***

(iv) (a) State Bertrand's Postulate. ***(2 marks)***

(b) Let p_n denote the n^{th} prime. Use Bertrand's Postulate to show that for $n \geq 8$, we have $p_n^2 < p_{n-1}p_{n-2}p_{n-3}$. ***(5 marks)***

(v) (a) Define the prime counting function $\pi(x)$ and state the Prime Number Theorem. ***(2 marks)***

(b) Evaluate $\lim_{x\to\infty} \dfrac{\pi(ax)}{\pi(x)}$ where a is a positive real number. ***(4 marks)***

2 (i) Given two arithmetic functions $f, g : \mathbb{N} \to \mathbb{C}$, give the definition of the Dirichlet convolution $f \star g : \mathbb{N} \mapsto \mathbb{C}$. ***(2 marks)***

(a) Consider the arithmetic functions $u, \Lambda : \mathbb{N} \to \mathbb{C}$ defined by

$$u(n) = 1 \quad \text{for every } n,$$
$$\Lambda(n) = \begin{cases} \ln(p), & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $(\Lambda \star u)(n) = \ln(n)$ for every $n \geq 1$. ***(4 marks)***

(b) Deduce the formal identity $D(s, \Lambda) = \dfrac{-\zeta'(s)}{\zeta(s)}$. ***(4 marks)***

(ii) Define the Riemann zeta function $\zeta(s)$ and write down the Euler product for $\zeta(s)$, indicating in what region of the complex plane they are valid. ***(4 marks)***

(iii) (a) Show, by estimating integrals or otherwise, that

$$\frac{1}{N^\sigma} + \int_1^N \frac{du}{u^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} < 1 + \int_1^N \frac{du}{u^\sigma}$$

for any integer $N > 1$ and real number $\sigma > 0$. ***(6 marks)***

(b) Using the inequality above, deduce that $\zeta(\sigma)$ diverges for real $\sigma \leq 1$, and converges for real $\sigma > 1$ and that $\zeta(\sigma)$ satisfies

$$\frac{1}{\sigma - 1} \leq \zeta(\sigma) \leq 1 + \frac{1}{\sigma - 1}$$

for real $\sigma > 1$. ***(6 marks)***

(iv) Let P be the arithmetic function defined by

$$P(n) = \begin{cases} 0 & \text{if } n = 1 \\ k & \text{if } n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k} \ (p_i\text{'s distinct}) \end{cases}$$

(a) Assuming that the function $2^{P(n)}$ is multiplicative, show that

$$D(s, 2^{P(n)}) = \frac{\zeta(s)^2}{\zeta(2s)}.$$

(6 marks)

(b) Using this deduce that

$$\sum_{\gcd(a,b)=1} \frac{1}{a^2 b^2} = \frac{\zeta(2)^2}{\zeta(4)}.$$

(3 marks)

3 This question asks you to illustrate the proof of Dirichlet's Theorem in a specific case.

(i) List the characters of $(\mathbb{Z}/18\mathbb{Z})^\star$. ***(6 marks)***

(ii) For all the non-trivial real-valued characters in your list, show explicitly that the corresponding Dirichlet L-series does not vanish at $s = 1$. ***(3 marks)***

(iii) Verify for all primes p that

$$\sum_\chi \chi(\overline{5})^{-1}\chi(\overline{p}) = \begin{cases} 6 & \text{if } p \equiv 5 \mod 18 \\ 0 & \text{otherwise} \end{cases}$$

where the sum runs over the characters of $(\mathbb{Z}/18\mathbb{Z})^\star$. ***(4 marks)***

(iv) Using the above, prove that there are infinitely many primes congruent to 5 modulo 18.
(You may assume that $L(\chi_i, 1) \neq 0$ for $i \neq 0$ and that for all χ, the sum $\sum_{p\neq 2,3}\sum_{n=2}^{\infty} \frac{\chi(p)}{np^{ns}}$ converges to a finite limit as $s \to 1$.) ***(8 marks)***

4 Bernouilli numbers B_k are defined by the generating series

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

(i) Write down, without proof, the numerical values of B_0, B_1, B_2 and B_k for odd $k > 1$. ***(3 marks)***

(ii) Prove the following identity for the Bernoulli numbers (for $k \geq 2$):

$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0.$$

(5 marks)

(iii) Compute B_4. ***(2 marks)***

End of Question Paper