The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2017-18**

Fields **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) State the Subfield Criterion. ***(4 marks)***

(ii) For each of the subsets J_1, J_2 of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$:

(a) $J_1 = \{a + b\sqrt{-2} : \ a, b \in \mathbb{Q}\}$, ***(5 marks)***

(b) $J_2 = \{a + b\sqrt{2} + ci : \ a, b, c \in \mathbb{Q}\}$. ***(3 marks)***

(iii) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $a = \sqrt{2} - 2\sqrt{3}$.

(a) Show that $L = \mathbb{Q}(a)$. ***(5 marks)***

(b) Express the element a^{-1} as a sum $\lambda_0 + \lambda_1\sqrt{2} + \lambda_2\sqrt{3} + \lambda_3\sqrt{6}$ where $\lambda_i \in \mathbb{Q}$. ***(3 marks)***

(c) Express the element $b = 2a^2\sqrt{2} - 4a\sqrt{3} + 10a - 10$ as a sum $\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3$ where $\alpha_i \in \mathbb{Q}$. ***(5 marks)***

2 Let $K \subseteq L$ be a field extension.

(i) What is meant by saying that an element $a \in L$ is *algebraic* over K? ***(2 marks)***

(ii) What is meant by saying that the field extension L is a *simple* field extension of K? ***(2 marks)***

(iii) Give a definition of the *minimal polynomial* $m(x) \in K[x]$ of the algebraic element a over K and prove that it is an irreducible polynomial over K. ***(6 marks)***

(iv) Suppose that $n = \deg(m(x))$. Show that $[K(a) : K] = n$ and find a K-basis the vector space $K(a)$ over the field K. ***(9 marks)***

(v) Find the minimal polynomial $m(x) \in \mathbb{Q}(\sqrt{3})[x]$ of the element $a = -\sqrt{2} + \sqrt{3}$ over the field $\mathbb{Q}(\sqrt{3})$. ***(6 marks)***

3 (i) Give the definition of a constructible number. ***(2 marks)***

(ii) Define a Fermat prime number. ***(2 marks)***

(iii) Let p be an odd prime number which is not a Fermat prime. Prove that the regular p-gon cannot be constructed. ***(8 marks)***

(iv) Prove that for all odd prime numbers p, the regular p^2-gon cannot be constructed. ***(7 marks)***

(v) Which of the following n-gons can be constructed $n = 60, 72, 165$? Justify your response. ***(6 marks)***

4 Let $K \subseteq L$ be fields.

(i) Define the group $G(L/K)$. ***(2 marks)***

(ii) Suppose that $L = K(a)$ for some element $a \in L$, $n = [L : K] < \infty$ and the field L contains all the roots, say $a_1 = a, a_2, \ldots, a_s$, of the minimal polynomial $m(x) \in K[x]$ of the element a over K. Describe explicitly the elements of the group $G(L/K)$ and show that $|G| = s \leq n$. ***(11 marks)***

(iii) Let $p \geq 3$ be a prime number, $a = e^{\frac{2\pi i}{p}}$, $K = \mathbb{Q}$ and $L = \mathbb{Q}(a)$.

(a) Write down the minimal polynomial $m(x)$ of the element a over $\mathbb{Q}$ (no proof is needed). ***(2 marks)***

(b) Describe explicitly all the elements of the group $G(L/\mathbb{Q})$. Justify your response. ***(5 marks)***

(iv) Let $\sigma \in G(L/K)$. Show that $L^{\sigma} = \{x \in L \mid \sigma(x) = x\}$ is a subfield of L that contains K. ***(5 marks)***

End of Question Paper