SCHOOL OF MATHEMATICS AND STATISTICS  

Fall Semester  
2017–18 

Financial Mathematics  

2 hours and 30 minutes 

Attempt all the questions. The allocation of marks is shown in brackets.
1. (i) A company borrows £100,000 paying a fixed annual interest rate of 5\% compounded continuously. The company will repay the loan in equal monthly payments.

(a) Find the amount of these payments which will repay the loan in 20 years. \(6\) marks

(b) Assume now that the company is given the option of repaying the loan with fixed monthly payments paid \textit{forever}. What is the amount of these payments in this case? \(4\) marks

(ii) Consider two risk-free, zero-coupon bonds with face value £100. The first matures in 1 year and trades for £99 and the second matures in 2 years and trades for £96.08.

(a) Find the 1-year and 2-year spot interest rates. \(2\) marks

(b) Find the forward interest rate for the period starting in 1 year and ending in 2 years. \(1\) mark

(c) You are offered an opportunity to borrow or deposit £100,000 for a period of one year starting 1 year hence at an interest rate of 2.5\%. Describe in detail an arbitrage opportunity available to you. \(12\) marks
(i) (a) Describe a portfolio consisting entirely of European call options on the same stock, with same expiration time $T > 0$, but with different strike prices, and whose payoff at time $T$ as a function of $S$, the spot price of the stock at time $T$, is described by the graph below. 

\[ \text{(6 marks)} \]

(b) Let $c_{10}$, $c_{20}$ and $c_{50}$ be the prices of the call options above with strike prices 10, 20 and 50, respectively, and let $p_{50}$ be the price of a European put option on the same stock, with expiration at time $T$ and with strike price 50. By comparing the payoff of the portfolio in (a) and the payoff of the put option above, describe an inequality involving $c_{10}, c_{20}, c_{50}$ and $p_{50}$. 

\[ \text{(6 marks)} \]

(ii) Consider two call options, one American and the other European, on the same underlying asset, with same expiration date $T$ and same strike price $X$. Let $S_t$ denote the underlying asset price at time $0 \leq t \leq T$. Let $C_t$ and $c_t$ denote the prices at time $0 \leq t \leq T$ of the American and European options, respectively. Assume all interest rates are constant and equal to $r$.

(a) By considering the payoffs at time $T$ of two portfolios consisting of the European call option, the underlying asset and cash, show that for all $0 \leq t \leq T$, $c_t \geq S_t - X e^{r(T-t)}$. 

\[ \text{(5 marks)} \]

(b) Suppose you observe that $C_0 > c_0$. Exhibit an arbitrage opportunity available to you. (You may find it useful to use (a) to deduce that for any $0 \leq t < T$, $c_t > S_t - X$.) 

\[ \text{(8 marks)} \]
(i) (a) State the mathematical definition of Brownian motion. \(5\) marks

(b) State Ito's Lemma. \(3\) marks

(ii) Assume that a stock price \(S\) is given as the Ito process

\[ dS = \mu S \, dt + \sigma S \, dB \]

where \(\mu\) and \(\sigma\) are constants. Assume also that all interest rates are non-stochastic and equal to \(r\). Let \(\Phi\) be the cumulative normal distribution function, i.e., \(\Phi(y) = \int_{-\infty}^{y} \phi(z)\,dz\) where \(\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\). Let \(X > 0\) and define

\[ d(S, t) = \frac{\log(S/X) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}. \]

Assume henceforth that \(f(S, t) = S\Phi(d(S, t))\) satisfies the Black-Scholes PDE.

\[ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \]

(a) Compute \(\lim_{t \to T, \ t<T} \Phi(d(S, t))\). (Hint: your answer should depend on the values of \(S\) and \(X\).) \(6\) marks

(b) Assume that \(f(S, t)\) is continuous for all \(0 \leq t \leq T\) and \(S \geq 0\). Compute \(f(0, t)\) for any \(0 \leq t < T\), and compute \(f(S_T, T)\) for any \(S_T \geq 0\). (Hint: use the assumption to write these values as limits.) \(5\) marks

(c) An asset-or-nothing option with strike price \(X\) and expiration time \(T\) pays owner at time \(T\): \(S_T\) if \(S_T > X\), \(S_T/2\) if \(S_T = X\), and nothing otherwise. Show that the price of this option at time \(0 \leq t \leq T\) is \(f(S_t, t)\) where \(S_t\) is the stock price at time \(t\). \(6\) marks
(i) Consider the following risky investments:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expected returns</th>
<th>Standard deviation of returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Inc.</td>
<td>9%</td>
<td>21%</td>
</tr>
<tr>
<td>B Plc.</td>
<td>5%</td>
<td>7%</td>
</tr>
<tr>
<td>C Ltd.</td>
<td>15%</td>
<td>36%</td>
</tr>
<tr>
<td>D S.A.</td>
<td>12%</td>
<td>15%</td>
</tr>
</tbody>
</table>

(a) Which investments cannot possibly be efficient? (4 marks)

(b) Suppose there is a risk-free return $R$ and you are told that C Ltd. is efficient, what can you say about the value of $R$? (8 marks)

(ii) Consider a world where there are only two risky investments: Krazy Plc and Stolid Inc. stocks.

<table>
<thead>
<tr>
<th>Number of shares</th>
<th>Price per share</th>
<th>Expected return</th>
<th>Standard deviation of return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Krazy Plc</td>
<td>£20</td>
<td>30%</td>
<td>60%</td>
</tr>
<tr>
<td>Stolid Inc.</td>
<td>£10</td>
<td>8%</td>
<td>10%</td>
</tr>
</tbody>
</table>

The correlation between the returns of these two stocks is 0.5.

(a) What is the market portfolio? (2 marks)

(b) What are the expected return and standard deviation of returns of the market portfolio? (3 marks)

(c) Find the beta coefficient of Krazy Plc. (4 marks)

(d) What should be the risk-free return in this world, if one existed? (4 marks)

End of Question Paper