SCHOOL OF MATHEMATICS AND STATISTICS  
Spring Semester 2017–2018

Algebra 1

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper \( K \) denotes a subfield of \( \mathbb{C} \) which contains \( \mathbb{Q} \).

All field extensions are finite.

1. (i) Find the roots of the cubic equation \( x^3 - 3x + 4 = 0 \). Simplify your answers as much as possible, making use of \( \omega \), a primitive cube root of 1, as needed. Identify the real root.

You may, if you wish, set \( x = u + v \) in terms of new variables \( u \) and \( v \), and proceed by obtaining equations for \( u^3 + v^3 \) and \( uv \).  

(ii) Show that \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3}) \).

Now let \( p \) and \( q \) be distinct primes. Show that \( \mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q}) \).

Find a polynomial \( f(x) \in \mathbb{Q}[x] \) such that \( \mathbb{Q}(\sqrt{p} + \sqrt{q}) \) is the splitting field of \( f(x) \). List the roots of \( f(x) \).
In each of (a), (b), (c), find the roots of the polynomial, and using them, determine the degree over \( \mathbb{Q} \) of the splitting fields. You may use, if you wish, a shifted Eisenstein criterion.

(a) \( x^4 + 1 \),

(b) \( x^6 + 1 \),

(c) \( x^6 + x^3 + 1 \).

(ii) Let \( G \) be a subgroup of a symmetric group \( S_n \), where \( n \geq 3 \). Define what it means for \( G \) to be transitive.

Prove that if \( n \) is prime and \( G \) is transitive and contains a transposition, then \( G = S_n \).

Let \( \xi = e^{2\pi i/11} \), and put

\[
\beta = \xi + \frac{1}{\xi} = 2 \cos \left( \frac{2\pi}{11} \right).
\]

(a) Show that \( \beta \) satisfies a quintic equation over \( \mathbb{Q} \), and write it down.

(b) Write \( \gamma = \xi + \xi^3 + \xi^4 + \xi^5 + \xi^9 \). Expand \( \gamma^2 \) in powers of \( \xi \), and hence deduce that \( \gamma^2 + \gamma + 3 = 0 \). Show that \( \mathbb{Q}(\sqrt{-11}) \subseteq \mathbb{Q}(\xi) \).

(c) State the general result(s) about the Galois groups of cyclotomic extensions from which it follows that \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \) is cyclic with 10 elements.

(d) Choose a generator \( \theta \) of \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \), being sure to define \( \theta \) explicitly. In terms of \( \theta \), write down the subgroups of \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \).

Let \( \xi \) be a primitive \( n \)th root of unity, where \( n \geq 3 \), and write \( \beta = \xi + \frac{1}{\xi} \).

(a) Show that \( \xi \) satisfies a quadratic equation over \( \mathbb{Q}(\beta) \) and deduce that \( [\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] \leq 2 \).

(b) Show that \( \mathbb{Q}(\beta) \subseteq \mathbb{R} \), and deduce that \( \xi \notin \mathbb{Q}(\beta) \).

(c) Deduce that \( [\mathbb{Q}(\xi) : \mathbb{Q}(\beta)] = 2 \) and hence calculate \( [\mathbb{Q}(\beta) : \mathbb{Q}] \), expressing your answer in terms of the \( \varphi \) function, defined so that \( \varphi(k) = |U(\mathbb{Z}_k)| \) for \( k \geq 2 \).
(i) State in full the Galois correspondence. Your answer should include the relation between indexes of subgroups and degrees of field extensions, and the significance of normal subgroups. (6 marks)

(ii) Define what it means for a group $G$ to be soluble. (3 marks)

(iii) Let $G$ be the dihedral group with 8 elements. Write the elements of $G$ as

$I, R, R^2, R^3, F, RF, R^2F, R^3F,$

where $R$ has order 4 and $F$ has order 2, and $FRF = R^{-1}$.

(a) Give the subgroup lattice of $G$ and deduce from it that $G$ is soluble. (12 marks)

(b) List the subgroups in the subgroup lattice that are not normal, giving brief justifications for your answers. (4 marks)

End of Question Paper