The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

MAS6352 Analysis II **2 hours 30 minutes**

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

Please leave this exam paper on your desk
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1 Throughout this question (S, Σ) is a measurable space, and $\mathbb{R}$ is equipped with its usual Borel σ–algebra $\mathcal{B}(\mathbb{R})$.

(i) Give three (distinct) equivalent formulations of what it means for $f : S \to \mathbb{R}$ to be measurable. ***(3 marks)***

(ii) Let f and g be measurable functions from S to $\mathbb{R}$.

(a) Prove that the set $\{f > g\} \in \Sigma$, where

$$\{f > g\} := \{x \in S; f(x) > g(x)\}.$$

(4 marks)

(b) Use (a) to show that $f - g$ is measurable. ***(3 marks)***

[Hint: You may use the fact that $g + a$ is measurable for all $a \in \mathbb{R}$, where $(g + a)(x) = g(x) + a$, for all $x \in \mathbb{R}$.]

(c) Deduce that $\{f = g\} \in \Sigma$, where

$$\{f = g\} := \{x \in S; f(x) = g(x)\}.$$

(4 marks)

[Hint: You may use the fact that $\{a\} \subseteq \mathcal{B}(\mathbb{R})$ for all $a \in \mathbb{R}$.]

(d) A *fixed point* of f is a solution of the equation $f(x) = x$. Show that the set of all fixed points of f is measurable. ***(2 marks)***

(iii) What can you say about the measurability of the set

$$E := \{x \in \mathbb{R}; \sqrt{1 + x^2}\sin(x + 3) = e^x\}?$$

Give brief arguments to support your conclusions. ***(2 marks)***

 Question 1 continued on next page

1 (continued)

(iv) Let (A_n) be an increasing sequence of sets in Σ and define the associated indicator functions in the usual way:

$$\mathbf{1}_{A_n}(x) = \begin{cases} 1 & \text{if } x \in A_n \\ 0 & \text{if } x \notin A_n \end{cases},$$

for all $n \in \mathbb{N}$.

(a) Show that $\lim\limits_{n\to\infty} \mathbf{1}_{A_n}(x)$ exists for all $x \in S$, and that the limit is a measurable function. ***(3 marks)***

(b) Construct a sequence (B_n) of mutually disjoint sets in Σ so that for all $n \in \mathbb{N}$,

$$\mathbf{1}_{A_n} = \sum_{r=1}^{n} \mathbf{1}_{B_r},$$

and use this to prove that $\mathbf{1}_A(x) = \lim\limits_{n\to\infty} \mathbf{1}_{A_n}(x)$ for all $x \in S$, where $A = \bigcup\limits_{n\in\mathbb{N}} A_n$. ***(5 marks)***

(v) We construct a variant on the Cantor set as follows. Start with the interval $[0,1]$ and remove the middle $1/5$ to obtain the set D_1. Then remove the middle $1/5$ of each of the disjoint intervals comprising D_1 to obtain D_2. Iterate this procedure to obtain a sequence of sets (D_n), and define $D = \bigcap\limits_{n=1}^{\infty} D_n$. Deduce a formula for the Lebesgue measure of D_n (there is no need to formally prove this) and hence obtain the Lebesgue measure of D, stating clearly any results you use to justify this last deduction. ***(7 marks)***

2 Throughout this question (S, Σ, m) is a fixed measure space.

(i) (a) Write down the general form of a non–negative *simple function*, explaining carefully the properties of any numbers and sets that appear in your expression. ***(3 marks)***

(b) Give a formula for the *Lebesgue integral* of the function in (a). ***(2 marks)***

(ii) Consider the function $g : \mathbb{R} \to \mathbb{R}$ that is given as follows:

$$g(x) = \begin{cases} -2 & \text{if } -5 \leq x < -3 \\ 1, & \text{if } -3 \leq x < -2, \\ 6 & \text{if } -2 \leq x < 1, \\ 4 & \text{if } 1 \leq x < 3, \\ -3 & \text{if } 3 \leq x < 5, \\ 0, & \text{if } |x| \geq 5 \end{cases}$$

(a) Write g explicitly as a simple function. ***(2 marks)***

(b) Define $h(x) = g(x) + 3$ for all $x \in \mathbb{R}$. Is h a simple function? If so write it explicitly. ***(3 marks)***

(c) Calculate $\displaystyle\int_{[-10,10]} g(x)dx$, $\displaystyle\int_{[-15,15]} h(x)dx$ and $\displaystyle\int_{\mathbb{R}} |g(x)|dx$. ***(8 marks)***

(iii) Let $f : S \to \mathbb{R}$ be an integrable function

(a) Prove that

$$\left| \int_S f(x)dm(x) \right| \leq \int_S |f(x)|dm(x),$$

making sure that you carefully introduce any tools that you need for the proof. ***(4 marks)***

(b) Explain why $\sin \circ f$ is a measurable function from S to $\mathbb{R}$, and deduce that

$$\left| \int_S \sin(f(x))dm(x) \right| < \infty.$$

(3 marks)

[Hint: You may use the inequality $|\sin(y)| \leq |y|$ for all $y \in \mathbb{R}$.]

(c) State the monotone convergence theorem, and use it to prove *Fatou's lemma*, i.e. given any sequence (f_n) of non–negative, measurable functions from S to $\mathbb{R}$ then

$$\liminf_{n \to \infty} \int_S f_n dm \geq \int_S \liminf_{n \to \infty} f_n dm.$$

(8 marks)

3 Throughout this question (S, Σ, m) is a fixed measure space.

(i) (a) Write down the definitions of both a π–system, and a λ–system. ***(4 marks)***

(b) If $\mathcal{L}$ is a λ–system, prove that

(α) If $A \in \mathcal{L}$ then $A^c \in \mathcal{L}$ ***(1 mark)***

(β) If (F_n) is a sequence in $\mathcal{L}$ with $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{L}$. ***(4 marks)***

(c) Define the following families of subsets of $\mathbb{R}$:

$$\mathcal{A}_1 := \{\emptyset, [a, b]; -\infty \leq a < b \leq \infty\},$$

$$\mathcal{A}_2 := \mathcal{A}_1 \cup \{\{c\}; c \in \mathbb{R}\}.$$

For each of $\mathcal{A}_1$ and $\mathcal{A}_2$, say whether it is a π–system, a λ–system, both or neither. Give reasons for your answers. ***(6 marks)***

(d) Prove that if $\mathcal{L}$ is both a π–system and a λ–system, then it is a σ–algebra. ***(4 marks)***

(ii) (a) State Fubini's theorem for integrable functions on a product space. ***(3 marks)***

(b) Let (f_n) be a sequence of measurable functions defined on S. In order to make rigorous sense of the quantity

$$I = \int_S \sum_{n=1}^{\infty} f_n(x) dm(x),$$

rewrite it using a suitable product measure, and then use Fubini's theorem to show that I is a real number if

$$\int_S \sum_{n=1}^{\infty} |f_n(x)| dm(x) < \infty.$$

(4 marks)

(c) Use the result of (b) to deduce that $\int_1^a \sum_{n=1}^{\infty} e^{-nx} f(x) dx$ exists for any $a > 0$, where f is a continuous function on $[1, a]$, and hence show that

$$\lim_{a\to\infty} \sum_{n=0}^{\infty} \int_1^a e^{-nx} dx = 1 - \log_e(e - 1).$$

(7 marks)

4 (i) (a) Let (S, Σ, m) be a measure space, and $f : S \to \mathbb{R}$ be a measurable function. Suppose that $|f|^n$ is integrable for some $n \in \mathbb{N}$. Prove the following generalisation of Markov's inequality:

$$m\{x \in S; |f(x)| > a\} \leq \frac{1}{a^n} \int_S |f|^n dm,$$

where $a > 0$. ***(4 marks)***

(b) Reformulate the result of (a) for a random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$, and having a finite nth moment. ***(2 marks)***

(c) We are given two random variables Y_1 and Y_2 defined on $(\Omega, \mathcal{F}, P)$. We are told that $\mathbb{E}(|Y_1|) = 12$ and $\mathbb{E}(|Y_2|^5) = 3$. We want to estimate $P(|Y_1| > 4)$ and $P(|Y_2| > 4)$ as accurately as possible. Is the inequality of (b) of any value in either of these cases? Present evidence, in the form of explicit calculations, to support your conclusions. ***(4 marks)***

(ii) Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Explain what it means for a sequence (X_n) of random variable to converge to a random variable X

(I) in mean square, (II) in probability, (III) almost surely.

(3 marks)

(a) Show that convergence in mean square implies convergence in probability. ***(2 marks)***

(b) State (without proof) any results that directly relate almost sure convergence to either of the other two types. ***(3 marks)***

(c) Let X be a fixed random variable for which $\mathbb{E}(|X|^2) < \infty$. Let (A_n) be a sequence of events in $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} A_n = \Omega$. Show that $(X\mathbf{1}_{A_n})$ converges in probability to a limit as $n \to \infty$, and find this limit. ***(7 marks)***

[Hint: $\lim_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1$, for all $\omega \in \Omega$.]

(iii) Let m be a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ having total mass M. Define the "h–function" $h : \mathbb{R} \to [0, \infty)$ by

$$h(x) = m((x, \infty)),$$

for all $x \in \mathbb{R}$.

(a) Show that h is monotonic decreasing. ***(2 marks)***

(b) Prove that $\lim_{x\to-\infty} h(x) = M$. ***(3 marks)***

[Hint: Take the limit using sequences.]

(c) If X is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, having cumulative distribution function F_X, show that the function $1 - F_X$ is an example of an h–function. ***(3 marks)***

End of Question Paper