Consider the set $\mathcal{L}_3$ of all oriented lines in $\mathbb{R}^3$ and the manifold $K_3 \subseteq \mathbb{R}^6$ defined by

$$K_3 = \{ (X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |X| = 1, \ X \cdot Y = 0 \}.$$ (8 marks)

(a) Describe, using neat diagrams as needed, the maps $\mathcal{L}_3 \to K_3$ and $K_3 \to \mathcal{L}_3$ which establish a bijection between $\mathcal{L}_3$ and $K_3$. (You are not asked to prove that these maps give a bijection.)

(b) Denote by $\Phi_L$ the map $\mathcal{L}_3 \to \mathcal{L}_3$ which sends each oriented line to the same line with the reverse orientation. Give a formula for the map $\Phi_K: K_3 \to K_3$ that corresponds to $\Phi_L$.

Write $U$ for the subset of $K_3$ which corresponds, under the above bijection, to the set of oriented lines that are tangent to the unit sphere. Determine $U$. (6 marks)

(c) Prove that $K_3$ is a manifold, and determine the tangent space $T_{(X,Y)}(K_3)$ at each $(X,Y) \in K_3$. State clearly any general theorem about smooth functions that you use. You are not asked to prove that the function(s) you use are smooth. (11 marks)
(i) Define what it means for a $2n \times 2n$ matrix $S$ to be symplectic. (2 marks)

(ii) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ matrix in block form, where $A, B, C$ and $D$ denote $n \times n$ matrices.

(a) Prove that $S$ is symplectic if and only if the three equations

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I,$$

hold.

(b) Assume that $S$ is symplectic and invertible. Give a formula for $S^{-1}$ in block form. (6 marks)

(iii) (a) Take $S \in Sp(4)$ and define a map $\Theta: (\mathbb{R}^4)^4 \to \mathbb{R}$ by

$$\Theta(Z_1, Z_2, Z_3, Z_4) = \Omega(Z_1, Z_2)\Omega(Z_3, Z_4) - \Omega(Z_1, Z_3)\Omega(Z_2, Z_4) + \Omega(Z_1, Z_4)\Omega(Z_2, Z_3),$$

where $Z_1, \ldots, Z_4 \in \mathbb{R}^4$. You may assume that $\Theta$ is multilinear. Show that it is skew-symmetric.

Using the following result, prove that $\det S = +1$.

Let $\Theta: (\mathbb{R}^4)^4 \to \mathbb{R}$ be a multilinear, skew-symmetric form such that $\Theta(E_1, E_2, E_3, E_4) = +1$, where $E_1, E_2, E_3, E_4$ is the standard basis of $\mathbb{R}^4$. Then $\Theta(Z_1, Z_2, Z_3, Z_4)$ is the determinant of the matrix with columns $Z_1, Z_2, Z_3, Z_4$, for any $Z_1, Z_2, Z_3, Z_4 \in \mathbb{R}^4$.

(b) Define a map $\Theta: (\mathbb{R}^6)^6 \to \mathbb{R}$ which may be expected to be multilinear, skew-symmetric, and have $\Theta(E_1, E_2, E_3, E_4, E_5, E_6) \neq 0$, where $E_1, E_2, E_3, E_4, E_5, E_6$ is the standard basis of $\mathbb{R}^6$. You are not asked to prove any of these properties. (17 marks)
(i) (a) Let \( S(t) \) be a symplectic matrix for all \( t \in \mathbb{R} \) with \( S(0) = I_{2n} \), the identity matrix.

Let \( H \) denote \( \frac{d}{dt} S(t) \bigg|_{t=0} \), the derivative of \( S(t) \) at \( t = 0 \).

Prove that
\[
H^T J + JH = 0. \tag{1}
\]

You may assume, without proof, that differentiation of matrix-valued functions obeys the usual rules for sums and products, with due attention to the order of factors in products.

(b) Now let \( H \) be a \( 2n \times 2n \) matrix which satisfies (1) and is such that \( I - H \) is invertible. Show that
\[
S = (I + H)(I - H)^{-1}
\]
is a symplectic matrix. (15 marks)

(ii) Let \( S \) be a \( 2n \times 2n \) matrix.

Denote the first \( n \) columns of \( S \) by \( F_1, \ldots, F_n \) and the last \( n \) columns by \( G_1, \ldots, G_n \). Prove any two of the following equations:

\[
\begin{align*}
\Omega(F_i, F_j) &= 0 & \text{for all } 1 \leq i, j \leq n, \\
\Omega(G_i, G_j) &= 0 & \text{for all } 1 \leq i, j \leq n, \\
\Omega(F_i, G_j) &= 0 & \text{for all } 1 \leq i, j \leq n, \quad i \neq j, \\
\Omega(F_i, G_i) &= 1 & \text{for all } 1 \leq i \leq n.
\end{align*}
\]

You may, if you wish, proceed by separating each \( F_i \) and \( G_j \) into pairs of elements of \( \mathbb{R}^n \). (10 marks)
(i) In Gaussian optics, refraction across a parabolic lens corresponds to a matrix of the form
\[
\begin{bmatrix}
1 & 0 \\
-m & 1
\end{bmatrix}
\]
where \(m \in \mathbb{R}\). Also in Gaussian optics, change of reference vertical corresponds to a matrix of the form
\[
\begin{bmatrix}
1 & w \\
0 & 1
\end{bmatrix}
\]
where we refer to \(w > 0\) as the optical distance.
(a) Calculate the matrix \(N\) corresponding to refraction through two parabolic lenses with coefficients \(m_1\) and \(m_2\), separated by optical distance \(w > 0\).
(b) Show that every symplectic matrix \(S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) with \(b \neq 0\) may be written as a matrix \(N\) as in (a) for unique values of \(m_1, m_2\) and \(w > 0\). (6 marks)

(ii) Let \(H_3\) denote the Heisenberg group, consisting of all matrices of the form
\[
\begin{bmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{bmatrix}
\]
for \(a, b, c \in \mathbb{R}\). Denote by \(h_3\) the vector space of all matrices of the form
\[
\begin{bmatrix}
0 & x & z \\
0 & 0 & y \\
0 & 0 & 0
\end{bmatrix}
\]
for \(x, y, z \in \mathbb{R}\).
Define the adjoint and coadjoint actions of \(H_3\) and calculate the orbits of each action. (19 marks)