The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Mathematics Core II **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: ***55***

Please leave this exam paper on your desk
Do not remove it from the hall

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Section A

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(22 marks)***

A1 Give the line perpendicular to $y = 2x + 1$ passing through $(0, 1)$.

A. $x + 2y = 2$ **B.** $2y = x + 2$ **C.** $y = 2x + 1$ **D.** $y = 2x - 1$

A2 Consider the point $P = (-1, \sqrt{3})$ in $\mathbb{R}^2$. What are the coordinates (r, θ) of P in polar coordinates?

A. $(2, 2\pi/3)$ **B.** $(-2, \pi/6)$ **C.** $(3, \pi/2)$ **D.** $(2, 4\pi/3)$

A3 Which of the following matrices is in row echelon form?

A. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 5 & 7 \end{pmatrix}$ **B.** $\begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$ **C.** $\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ **D.** $\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 1 \end{pmatrix}$

A4 A system of three equations in the 5 variables x, y, z, u and v are reduced to a system in reduced row echelon for $\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{array}\right)$. Which variables are the easiest to use as parameters?

A. x, y and z **B.** x, y and u **C.** z and u **D.** z and v

A5 A is a 2×3 matrix, B is a 3×1 matrix and C is a 2×3 matrix. Which of the following products make sense?

A. AB **B.** AC **C.** BC **D.** CA

A6 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, what is AB?

A. $\begin{pmatrix} 1 & -2 \\ 6 & 2 \end{pmatrix}$ **B.** $\begin{pmatrix} -2 & 0 \\ 5 & 6 \end{pmatrix}$ **C.** $\begin{pmatrix} -1 & 4 \\ 1 & 8 \end{pmatrix}$ **D.** $\begin{pmatrix} 5 & 1 \\ 7 & -1 \end{pmatrix}$

A7 What is the inverse matrix of $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$?

A. $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ **B.** $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ **C.** $\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$ **D.** $\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$

A8 What is the determinant of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & a \end{pmatrix}$?

A. $-3+3a$ **B.** $a+2$ **C.** $12-3a$ **D.** $13-a$

A9 The numbers a, b and c are such that $\begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = 4$. Let $d_1 = \begin{vmatrix} 2 & 1 & 0 \\ a & b & c \\ -1 & 1 & 1 \end{vmatrix}$, $d_2 = \begin{vmatrix} a & b & 2c \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix}$, and $d_3 = \begin{vmatrix} a+2 & b+1 & c \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix}$. Which of the following holds?

A. $d_1 > d_2 > d_3$ **B.** $d_2 > d_3 > d_1$ **C.** $d_3 > d_2 > d_1$ **D.** $d_3 > d_1 > d_2$

A10 The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. What is the corresponding eigenvalue?

A. -1 **B.** 1 **C.** $\dfrac{1+\sqrt{5}}{2}$ **D.** 4

A11 Write down the characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$.

A. $\lambda^2 - 14\lambda + 1$ **B.** $\lambda^2 + 9\lambda + 9$ **C.** $\lambda^2 - 9\lambda - 1$ **D.** None of the above

A12 What is the sum of the eigenvalues of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \end{pmatrix}$?

A. 1 **B.** 2 **C.** 3 **D.** 4

A13 Consider the function $f(x,y) = e^{xy+y^2}$. What is $\dfrac{\partial f}{\partial x}$?

A. e^x **B.** ye^x **C.** e^{xy+y^2} **D.** ye^{xy+y^2}

A14 What is the tangent plane to $z = x^2 + y^2$ at the point $(1, -2, 5)$?

A. $z = 2x - 4y - 5$ **B.** $2x + 2y + z = 3$ **C.** $x - 2y + 5z = 0$ **D.** $z = 2x + 2y$

A15 Recall that $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$. What is $\cosh^2 x + \sinh^2 x$?

A. 1 **B.** e^x **C.** $\cosh 2x$ **D.** e^{x^2}

A16 The curve given by the quadratic formula $2x^2 + 3y^2 - 12x - 12y + 5 = 0$ is

A. a circle **B.** a parabola **C.** an ellipse **D.** a hyperbola

A17 One eigenvector of $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is $\begin{pmatrix} \sqrt{2}-1 \\ 1 \end{pmatrix}$. Which of these is an example of an eigenvector corresponding to the other eigenvalue?

A. $\begin{pmatrix} \sqrt{2}+1 \\ 1 \end{pmatrix}$ **B.** $\begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$ **C.** $\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$ **D.** $\begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$

A18 You may assume that the eigenvalues of $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are 1 and 5. There is a stationary point of $z = 3x^2 + 4xy + 3y^2$ at $(0, 0, 0)$. This point is

A. a maximum **B.** a minimum **C.** a saddle point **D.** none of these

A19 The function $I(x)$ is $\int_0^{2x} e^{3t}\, dt$. What is $\frac{dI}{dx}$?

A. $6e^x$ **B.** $2e^{3x}$ **C.** $2e^{6x}$ **D.** $6e^{2x}$

A20 The triangle T has vertices at $(-1, 1)$, $(1, 1)$ and $(1, 0)$. We can compute its area with

A. $\int_{-1}^{1}\int_{0}^{1} 1\, dx\, dy$ **B.** $\int_{0}^{1}\int_{1-2y}^{1} 1\, dx\, dy$ **C.** $\int_{-1}^{1}\int_{0}^{1} 1\, dy\, dx$ **D.** $\int_{1-2y}^{1}\int_{0}^{1} 1\, dy\, dx$

A21 What is $\int_0^1 \int_0^y xy\, dx\, dy$?

A. $\frac{1}{8}$ **B.** $\frac{1}{6}$ **C.** $\frac{1}{4}$ **D.** $\frac{1}{2}$

A22 Write $x = u^2 + v^2$ and $y = uv$. Compute $\frac{\partial(x, y)}{\partial(u, v)}$.

A. $u^2 + v^2$ **B.** $u + v$ **C.** $2uv$ **D.** $2u^2 - 2v^2$

Section B

B1 Give the general solution to the system of equations:

$$\begin{aligned} x + 2y + 3z &= -1 \\ -x + y + 2z &= 3 \\ 4x - y - 3z &= -10. \end{aligned}$$

Geometrically, what is the set of solutions? ***(4 marks)***

B2 Find the unique value of λ such that the vector $(6, 9, \lambda)$ is a linear combination of $(1, 2, 3)$, $(2, 1, -1)$ and $(0, 3, 7)$.

The three vectors $(1, 2, 3)$, $(2, 1, -1)$ and $(0, 3, 7)$ are the normal vectors to planes passing through the origin $(0, 0, 0)$. What can you say about how these three planes intersect? ***(3 marks)***

B3 Consider a 2×2-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(i) What is the condition for A to be invertible?

(ii) When A is invertible, write down its inverse A^{-1} and show that $\det A^{-1} = 1/\det A$. ***(2 marks)***

B4 Show that the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ has a double root, and find two eigenvectors with this root as eigenvalue which are orthogonal. ***(3 marks)***

B5 Consider the function $z = e^{xy}\cos(x + y)$. By computing its partial derivatives, give the Taylor series around $(0, 0)$ as far as the degree 2 terms. ***(5 marks)***

B6 Let $f(x, y) = \sin(xy)$. Evaluate f at the point $(\frac{\pi}{2}, 0)$. Using partial derivatives, approximate the change δf when we move to the nearby point $(\frac{\pi}{2} + 0.1, -0.2)$. ***(3 marks)***

B7 Find the stationary points of the function $f(x, y) = 6xy - 2x^3 - 3y^2$, and determine the nature of each stationary point. ***(4 marks)***

B8 Find the area of the surface obtained by rotating the graph of $f(x) = \cosh x$ about the x-axis over the interval $[0, 1]$. ***(4 marks)***

B9 By switching to polar coordinates, calculate the integral of the function $f(x, y) = xy$ over the region $D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$.

You may quote any standard result for change of coordinates from Cartesian to polar coordinates. ***(5 marks)***

End of Question Paper