SCHOOL OF MATHEMATICS AND STATISTICS  
Spring Semester 2017–2018

Mathematics Core II  
2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: 55

Please leave this exam paper on your desk  
Do not remove it from the hall

Registration number from U-Card (9 digits) to be completed by student
Section A

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. (22 marks)

A1 Give the line perpendicular to $y = 2x + 1$ passing through $(0, 1)$.
   
   A. $x + 2y = 2$  
   B. $2y = x + 2$  
   C. $y = 2x + 1$  
   D. $y = 2x - 1$

A2 Consider the point $P = (-1, \sqrt{3})$ in $\mathbb{R}^2$. What are the coordinates $(r, \theta)$ of $P$ in polar coordinates?
   
   A. $(2, 2\pi/3)$  
   B. $(-2, \pi/6)$  
   C. $(3, \pi/2)$  
   D. $(2, 4\pi/3)$

A3 Which of the following matrices is in row echelon form?

   A. $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 5 & 7 \end{pmatrix}$  
   B. $\begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$  
   C. $\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$  
   D. $\begin{pmatrix} 0 & 1 & 2 & 4 \\ 0 & 2 & 3 & 1 \end{pmatrix}$

A4 A system of three equations in the 5 variables $x$, $y$, $z$, $u$ and $v$ are reduced to a system in reduced row echelon form for $\begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 0 \end{pmatrix}$. Which variables are the easiest to use as parameters?

   A. $x$, $y$ and $z$  
   B. $x$, $y$ and $u$  
   C. $z$ and $u$  
   D. $z$ and $v$

A5 $A$ is a $2 \times 3$ matrix, $B$ is a $3 \times 1$ matrix and $C$ is a $2 \times 3$ matrix. Which of the following products make sense?

   A. $AB$  
   B. $AC$  
   C. $BC$  
   D. $CA$

A6 If $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, what is $AB$?

   A. $\begin{pmatrix} 1 & -2 \\ 6 & 2 \end{pmatrix}$  
   B. $\begin{pmatrix} -2 & 0 \\ 5 & 6 \end{pmatrix}$  
   C. $\begin{pmatrix} -1 & 4 \\ 1 & 8 \end{pmatrix}$  
   D. $\begin{pmatrix} 5 & 1 \\ 7 & -1 \end{pmatrix}$

A7 What is the inverse matrix of $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$?

   A. $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$  
   B. $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$  
   C. $\begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$  
   D. $\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$
A8 What is the determinant of \(
\begin{pmatrix}
1 & 2 & 3 \\
1 & -1 & 1 \\
2 & 1 & a
\end{pmatrix}
\)?

A. \(-3 + 3a\)  
B. \(a + 2\)  
C. \(12 - 3a\)  
D. \(13 - a\)

A9 The numbers a, b and c are such that \(\begin{vmatrix}
a & b & c \\
2 & 1 & 0 \\
-1 & 1 & 1
\end{vmatrix} = 4\). Let \(d_1 = \begin{vmatrix}
a & b \\
-1 & 1
\end{vmatrix}\), \(d_2 = \begin{vmatrix}
a & b & 2c \\
2 & 1 & 0 \\
-1 & 1 & 2
\end{vmatrix}\), and \(d_3 = \begin{vmatrix}
a + 2 & b + 1 & c \\
2 & 1 & 0 \\
-1 & 1 & 1
\end{vmatrix}\). Which of the following holds?

A. \(d_1 > d_2 > d_3\)  
B. \(d_2 > d_3 > d_1\)  
C. \(d_3 > d_2 > d_1\)  
D. \(d_3 > d_1 > d_2\)

A10 The vector \(\begin{pmatrix}1 \\ 2 \end{pmatrix}\) is an eigenvector of \(\begin{pmatrix}2 & 1 \\ 2 & 3 \end{pmatrix}\). What is the corresponding eigenvalue?

A. \(-1\)  
B. \(1\)  
C. \(\frac{1 + \sqrt{5}}{2}\)  
D. \(4\)

A11 Write down the characteristic polynomial of \(\begin{pmatrix}2 & 3 \\ 5 & 7 \end{pmatrix}\).

A. \(\lambda^2 - 14\lambda + 1\)  
B. \(\lambda^2 + 9\lambda + 9\)  
C. \(\lambda^2 - 9\lambda - 1\)  
D. None of the above

A12 What is the sum of the eigenvalues of \(\begin{pmatrix}1 & 0 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \end{pmatrix}\)?

A. \(1\)  
B. \(2\)  
C. \(3\)  
D. \(4\)

A13 Consider the function \(f(x, y) = e^{xy+y^2}\). What is \(\frac{\partial f}{\partial x} \)?

A. \(e^x\)  
B. \(ye^x\)  
C. \(e^{xy+y^2}\)  
D. \(ye^{xy+y^2}\)

A14 What is the tangent plane to \(z = x^2 + y^2\) at the point \((1, -2, 5)\)?

A. \(z = 2x - 4y - 5\)  
B. \(2x + 2y + z = 3\)  
C. \(x - 2y + 5z = 0\)  
D. \(z = 2x + 2y\)

A15 Recall that \(\cosh x = \frac{e^x + e^{-x}}{2}\) and \(\sinh x = \frac{e^x - e^{-x}}{2}\). What is \(\cosh^2 x + \sinh^2 x\)?

A. \(1\)  
B. \(e^x\)  
C. \(\cosh 2x\)  
D. \(e^{x^2}\)
A16 The curve given by the quadratic formula $2x^2 + 3y^2 - 12x - 12y + 5 = 0$ is
A. a circle B. a parabola C. an ellipse D. a hyperbola

A17 One eigenvector of $\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ is $\begin{pmatrix} \sqrt{2} - 1 \\ 0 \end{pmatrix}$. Which of these is an example of an eigenvector corresponding to the other eigenvalue?
A. $\begin{pmatrix} \sqrt{2} + 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ 1 - \sqrt{2} \end{pmatrix}$ C. $\begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 1 \\ -1 - \sqrt{2} \end{pmatrix}$

A18 You may assume that the eigenvalues of $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are 1 and 5. There is a stationary point of $z = 3x^2 + 4xy + 3y^2$ at $(0, 0, 0)$. This point is
A. a maximum B. a minimum C. a saddle point D. none of these

A19 The function $I(x)$ is $\int_0^{2x} e^{3t} dt$. What is $\frac{dI}{dx}$?
A. $6e^x$ B. $2e^{3x}$ C. $2e^{6x}$ D. $6e^{2x}$

A20 The triangle $T$ has vertices at $(-1, 1), (1, 1)$ and $(1, 0)$. We can compute its area with
A. $\int_{-1}^1 \int_0^1 dx\,dy$ B. $\int_0^1 \int_{-2}^1 dx\,dy$ C. $\int_{-1}^1 \int_0^1 dy\,dx$ D. $\int_{-2}^1 \int_0^1 dy\,dx$

A21 What is $\int_0^1 \int_0^y xy\,dx\,dy$?
A. $\frac{1}{8}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{2}$

A22 Write $x = u^2 + v^2$ and $y = uv$. Compute $\frac{\partial (x, y)}{\partial (u, v)}$.
A. $u^2 + v^2$ B. $u + v$ C. $2uv$ D. $2u^2 - 2v^2$
Section B

B1  Give the general solution to the system of equations:

\[
\begin{align*}
    x + 2y + 3z &= -1 \\
    -x + y + 2z &= 3 \\
    4x - y - 3z &= -10.
\end{align*}
\]

Geometrically, what is the set of solutions?  \((4\text{ marks})\)

B2  Find the unique value of \(\lambda\) such that the vector \((6, 9, \lambda)\) is a linear combination of \((1, 2, 3)\), \((2, 1, -1)\) and \((0, 3, 7)\).

The three vectors \((1, 2, 3)\), \((2, 1, -1)\) and \((0, 3, 7)\) are the normal vectors to planes passing through the origin \((0, 0, 0)\). What can you say about how these three planes intersect?  \((3\text{ marks})\)

B3  Consider a \(2 \times 2\)-matrix \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\).

(i)  What is the condition for \(A\) to be invertible?

(ii)  When \(A\) is invertible, write down its inverse \(A^{-1}\) and show that \(\det A^{-1} = 1 / \det A\).  \((2\text{ marks})\)

B4  Show that the characteristic polynomial of the matrix \(\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\) has a double root, and find two eigenvectors with this root as eigenvalue which are orthogonal.  \((3\text{ marks})\)

B5  Consider the function \(z = e^{xy} \cos(x + y)\). By computing its partial derivatives, give the Taylor series around \((0, 0)\) as far as the degree 2 terms.  \((5\text{ marks})\)

B6  Let \(f(x, y) = \sin(xy)\). Evaluate \(f\) at the point \((\pi, 0)\). Using partial derivatives, approximate the change \(\delta f\) when we move to the nearby point \((\pi + 0.1, -0.2)\).  \((3\text{ marks})\)

B7  Find the stationary points of the function \(f(x, y) = 6xy - 2x^3 - 3y^2\), and determine the nature of each stationary point.  \((4\text{ marks})\)

B8  Find the area of the surface obtained by rotating the graph of \(f(x) = \cosh x\) about the \(x\)-axis over the interval \([0, 1]\).  \((4\text{ marks})\)

B9  By switching to polar coordinates, calculate the integral of the function \(f(x, y) = xy\) over the region \(D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}\).

You may quote any standard result for change of coordinates from Cartesian to polar coordinates.  \((5\text{ marks})\)

End of Question Paper