The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Numbers and Groups **2 hours**

Answer ***all questions****.*
You should justify your answers carefully unless the question states otherwise.

1 We define a sequence $a_0, a_1, \ldots$ by the formulae $a_0 = 2$, $a_1 = 2$, and

$$a_{n+2} = a_{n+1}^2 + a_n + 1$$

for $n \geq 0$.

(i) What are a_2 and a_3? ***(2 marks)***

(ii) Show that a_n is two more than a multiple of five for all n. ***(4 marks)***

(iii) Show that $a_n \geq 2^{2^{n-1}}$ for all $n \geq 1$. ***(4 marks)***

2 (i) Which of the following implications are true for *all* real numbers x? You do not have to justify your answers.

(a) If $x = 4$, then $x^2 - 7x + 12 = 0$.

(b) If $x^2 - 7x + 12 = 0$, then $x = 4$.

(c) If $x^2 - 7x + 6 = 0$, then $x = 4$.

(d) If $x^2 + 13 = 0$, then $x = 4$. ***(4 marks)***

(ii) (a) State what it means for a sequence $a_1, a_2, \ldots$ to converge to a number x. ***(2 marks)***

(b) Give an example of a number N for which $1/\log\log(x) < 0.001$ whenever $x > N$. ***(1 mark)***

(c) Show that the sequence $a_n = 1/\log\log(n+1)$ converges, stating clearly what its limit is. ***(3 marks)***

3 (i) Find all n for which $33n \equiv 12 \pmod{501}$, justifying your working. ***(5 marks)***

(ii) (a) State Fermat's Little Theorem. ***(2 marks)***

(b) Show that, for any prime p between 10 and 100, we have

$$10^{100!} \equiv 1 \pmod{p}.$$

(2 marks)

(c) Give an example of a number n such that $10^{100!}$ is not congruent to 1 modulo n. ***(1 mark)***

4 Let $n \geq 2$ and, as usual, let S_n denote the set of all permutations of the numbers $1, \ldots, n$ (that is, the set all bijective functions $\{1, \ldots, n\} \to \{1, \ldots, n\}$).

(i) Show that S_n satisfies the group axioms under composition.

(You may use the fact that composition of functions is always associative.) ***(4 marks)***

(ii) For a permutation $\alpha \in S_n$, define what is meant by the *sign* of α. ***(2 marks)***

(iii) Show that the set $A_n = \{\alpha \in S_n : \operatorname{sgn}(\alpha) = 1\}$ is a subgroup of S_n. ***(4 marks)***

5 (i) In S_6, let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}.$$

(a) Find the cycle decomposition of α and state the order of α. ***(2 marks)***

(b) Show that if $\gamma \in S_6$ is such that $\gamma^2 = \alpha$ then the order of γ is either 3 or 6. Find an example of such a γ with each of these orders. ***(4 marks)***

(ii) Let G be a group of order 30. For each of the statements below, say whether it is true or false, justifying your answer.

(a) G has no subgroup of order 7;

(b) Lagrange's Theorem implies that G has a subgroup of order 5;

(c) Any subgroup of G of order 5 is cyclic;

(d) Any subgroup of G of order 5 is abelian. ***(4 marks)***

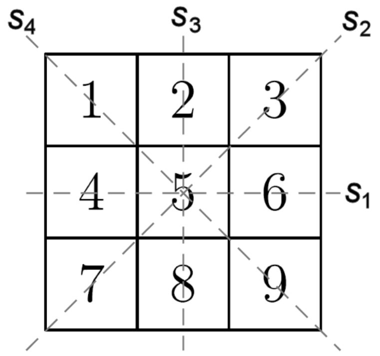
6 (i) The group $D_4 = \{e,\ r_1,\ r_2,\ r_3,\ s_1,\ s_2,\ s_3,\ s_4\}$ acts on the numbered regions 1 to 9 below in the usual way, where $r_i = \mathrm{rot}_{\frac{\pi i}{2}}$ for each i and $s_1,\ s_2,\ s_3$ and s_4 are reflections in the lines indicated below. For this action, write down the elements in

(a) $\mathrm{orb}(1)$;

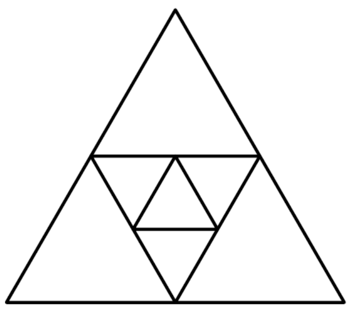
(b) $\mathrm{stab}(4)$;

(c) $\mathrm{fix}(r_3)$;

(d) $\mathrm{send}_2(8)$. ***(4 marks)***



(ii) Find the number of essentially different ways the equilateral triangular tile below, which can be turned over, can be coloured so that there are 4 blue regions and 3 green regions, making your workings clear. ***(6 marks)***



End of Question Paper