The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

MAS140 Mathematics (Chemical) **3 hours**

*Attempt **ALL** the questions.*
Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.
All solutions should be justified in full. Calculators should be relied upon for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \dfrac{3x+2}{3-x} + 2$. State the domain of $f(x)$. Find $f^{-1}(x)$ and give the domain of $f^{-1}(x)$.

A2 Find $\dfrac{d}{dx}\left(\dfrac{x^2 e^x}{x+1}\right)$.

A3 Evaluate

$$\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$$

by using Maclaurin series expansions, or l'Hospital's Rule.

A4 Derive Euler's formula

$$e^{ix} = \cos x + i \sin x$$

by using Taylor series expansions.

A5 Find the locus of the complex numbers z that satisfy $|z+2i-3| = |z+3i|$. Find also the complex number that lies on this locus and has $\arg z = \pi/4$.

A6 Determine a and b such that

$$(a\boldsymbol{i} + b\boldsymbol{j} + \boldsymbol{k}) \times (2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}) = \boldsymbol{i} - \boldsymbol{j}$$

holds, where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the standard unit vectors.

A7 Find the indefinite integral $\int \sqrt{x^2-9}\,dx$ by the substitution $x = 3\cosh u$.

A8 Find the indefinite integral $\int \frac{dx}{x\ln x}$ by using a suitable substitution.

A9 Solve the following system of equations

$$\begin{cases} x - y + z = 0, \\ 3x + y - 4z = 0, \\ 5x - y - 2z = 0. \end{cases}$$

A10 Find and classify the stationary points of $f(x,y) = 3x^2 + xy + 2y$.

A11 Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find $A^T A$ and AA^T, if it is possible to define either or both. Here A^T denotes the transpose of matrix A.

A12 Find the general solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$$

Section B

B1 (i) Consider the definite integrals

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx, \;\; J = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx.$$

Show that $J - I = 0$.

(ii) Compute $I + J$, hence evaluate I.

B2 Consider the following differential equation

$$\frac{dy}{dt} + y = e^{-at},$$

where a is a constant and $y(0) = 0$.

(i) Solve it for $a \neq 1$.

(ii) By taking the limit $a \to 1$, or otherwise, solve it for $a = 1$.

B3 Let $A = \begin{pmatrix} 6 & -3 & -7 \\ -1 & 2 & 1 \\ 5 & -3 & -6 \end{pmatrix}$. Find all the eigenvalues of A and the corresponding eigenvectors.

B4 (i) Show that the Laplace transform $F(s)$ of $f(t) = \dfrac{\sin t}{t}$ satisfies

$$F'(s) = -\frac{1}{s^2+1}.$$

(ii) Derive

$$F(s) = -\tan^{-1} s + \frac{\pi}{2}$$

and hence show that

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

B5 The position vector $\boldsymbol{r}(t)$ of a particle is given by

$$\boldsymbol{r}(t) = 2(\cos t)\boldsymbol{i} + 2(\sin t)\boldsymbol{j} + e^{-t}\boldsymbol{k},$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the standard unit vectors.

(i) Find the velocity $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$ and the acceleration $\boldsymbol{a} = \dfrac{d^2\boldsymbol{r}}{dt^2}$ at time t.

(ii) When $t = 0$, find a unit vector $\boldsymbol{t}$ in the direction of $\boldsymbol{v}$. Also, find the tangential component of $\boldsymbol{a}$ in the direction of $\boldsymbol{v}$.

B6 (i) By differentiating the right-hand side, or otherwise, show that

$$\int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{(1+x^2)^{1/2}} + C$$

holds.

(ii) Solve the linear differential equation

$$(1+x^2)\frac{dy}{dx} = xy + 1.$$

Hint: You may make use of (i).

B7 Show that

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{2^{n/2} \cos \frac{n\pi}{4}}{n!} x^n.$$

Hint: By $\cos x = \operatorname{Re}\left(e^{ix}\right)$, we may regard the left-hand side as the real part of a complex function.

B8 Using Gaussian elimination, or otherwise, solve the following system of linear equations

$$\begin{cases} 2x - y + 3z = -3, \\ x + 2y - 2z = 7, \\ 4x + 3y + z = 9. \end{cases}$$

End of Question Paper

MAS140/151/152/156/161 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \mathrm{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x + y) + \cos(x - y)$

$2\sin x \sin y = \cos(x - y) - \cos(x + y)$

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2\tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\mathrm{sech} x = \frac{1}{\cosh x}$$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\mathrm{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1$$

Series

Sum of an arithmetic series:

$$\frac{\text{first term } + \text{ last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r} x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)$$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\frac{1}{\cosh^2 x}$	$\coth x$	$-\frac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\frac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a > 0,\ a \neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x$$

$$\int \operatorname{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \frac{1}{a}\tanh^{-1}\frac{x}{a} \quad \text{if } |x| < a\right)$$

$$\int \operatorname{cosec} x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln\left(\operatorname{cosec} x - \cot x\right)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \quad \text{or} \quad \ln\left(\sec x + \tan x\right)$$

$$\int \operatorname{cosech} x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

Integration by parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha,\, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\dfrac{1}{s-a}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$