The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

Please leave this exam paper on your desk
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1 (i) Given a group G, what does it mean for it to be abelian? ***(1 mark)***

(ii) In the group S_5 of permutations of the set $\{1, 2, 3, 4, 5\}$, write down (as a product of disjoint cycles), the element $\beta\alpha\beta^{-1}$, where $\alpha = (1\,2)(3\,4\,5)$ and $\beta = (1\,3)(2\,4)$. ***(1 mark)***

(iii) In the group S_7, let $\alpha = (1\,2)$. Write down the orders of S_7, $\mathrm{conj}_{S_7}(\alpha)$ and $\mathrm{cent}_{S_7}(\alpha)$. ***(3 marks)***

(iv) In the group $\mathbb{Z}$ of integers under addition, write down all the elements of the conjugacy class of 3. ***(1 mark)***

(v) Write $x^3 - 1$ as a product of irreducible elements in the rings

(a) $\mathbb{Q}[x]$; ***(1 mark)***

(b) $\mathbb{R}[x]$; ***(1 mark)***

(c) $\mathbb{C}[x]$. ***(1 mark)***

(vi) In the inner product space $\mathbb{R}^4$ with dot product, calculate the angle θ between the vectors $\mathbf{a} = \begin{pmatrix} \sqrt{3} \\ 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ \sqrt{3} \end{pmatrix}$. ***(2 marks)***

(vii) Is the group $\mathbb{C}^{\times}$, of non-zero complex numbers under multiplication, abelian? (Yes or No.) ***(1 mark)***

(viii) Is the set of non-zero, two-by-two matrices with real entries, a group under the operation of matrix multiplication? (Yes or No.) ***(1 mark)***

(ix) Is the element 101 of the ring of Gaussian integers $\mathbb{Z}[i]$ irreducible? (Yes or No.) ***(1 mark)***

(x) Is $\mathbb{Z}[x]$ an integral domain? (Yes or No.) ***(1 mark)***

(xi) Is $\mathbb{R}$ a field? (Yes or No.) ***(1 mark)***

(xii) Is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$? (Yes or No.) ***(1 mark)***

2 Let S_3 be the group of permutations of the set $\{1, 2, 3\}$, under the operation of composition. You may assume that $H := \{\mathrm{id}, (12)\}$ is a subgroup of S_3. Write out the left cosets of H, including H itself, listing between curly brackets the elements of each coset, each element written as a product of disjoint cycles. Similarly write down the right cosets of H. Is H a normal subgroup of S_3? Provide some brief justification of your answer. ***(3 marks)***

3 Let $\mathbb{C}^\times$ be the group of non-zero complex numbers under multiplication. Consider the map $f : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $f(z) = z^4$.

(i) Prove that f is a group homomorphism. ***(1 mark)***

(ii) List the elements of the kernel of f. ***(1 mark)***

(iii) List all the elements of $f^{-1}(\{16\})$. Is f an isomorphism? (Yes or No.) ***(2 marks)***

4 Consider the map $\theta : \mathbb{C} \to M_2(\mathbb{R})$ defined by $\theta(a+bi) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{R}$. Prove that θ is a ring homomorphism. Give two independent justifications for the statement that it is not a ring isomorphism between $\mathbb{C}$ and $M_2(\mathbb{R})$. (You could try to show that it is not a bijection. You could also consider that any "structural" property of a ring is shared by any other ring isomorphic to it.) ***(5 marks)***

5 Why is the ring $\mathbb{Z}/71\mathbb{Z}$ a field? Find the multiplicative inverse of the element $\overline{11}$ of $\mathbb{Z}/71\mathbb{Z}$. ***(2 marks)***

6 (i) What does it mean for a function $f : V \to W$, where V and W are F-vector spaces, F a field, to be a *linear map*? Prove that the kernel, $\ker(f)$, is a subspace of V. (You may assume that $f(0_V) = 0_W$, and that $\lambda\, 0_W = 0_W$ for any $\lambda \in F$.) ***(4 marks)***

(ii) By solving a set of linear equations, find a basis for the kernel of the linear map $\ell : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $\ell(\mathbf{x}) := A\mathbf{x}$, where $A := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}$. ***(2 marks)***

(iii) Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, the $\mathbb{R}$-vector space of real-valued functions, with derivatives of all orders, of a real variable. Let $L(V)$ be the ring of linear operators on V, and consider $D \in L(V)$ defined by $D(y) := \dfrac{dy}{dx}$. By solving a homogeneous second-order differential equation, find a basis for the subspace $\ker(D^2 - 4D + 4)$ of V (where "4" means multiplication by 4, so $4(y) = 4y$). ***(2 marks)***

7 Let $\mathbb{F}_5 := \mathbb{Z}/5\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ be the field with 5 elements. For convenience we may write "a" in place of "$\overline{a}$". Let $\mathbb{F}_5[x]$ be the set of polynomials in one variable with coefficients in $\mathbb{F}_5$, considered as an $\mathbb{F}_5$-vector space. For any integer $n \geq 0$ there is a subspace $\mathbb{F}_5[x]_{\leq n}$ of polynomials of degree n or less.

(i) Write down the dimension of the $\mathbb{F}_5$-vector space $\mathbb{F}_5[x]_{\leq n}$. ***(1 mark)***

(ii) Let $d_1 : \mathbb{F}_5[x]_{\leq 3} \to \mathbb{F}_5[x]_{\leq 3}$ be an $\mathbb{F}_5$-linear map with the property that $d_1(x^n) = nx^{n-1}$. What is $d_1(x^3 + 2x^2 + x + 3)$? Write down bases for the kernel and image of d_1. What are the rank and the nullity of d_1? ***(4 marks)***

(iii) Let $d_2 : \mathbb{F}_5[x]_{\leq 5} \to \mathbb{F}_5[x]_{\leq 5}$ be an $\mathbb{F}_5$-linear map with the property that $d_2(x^n) = nx^{n-1}$. Write down bases for the kernel and image of d_2. ***(2 marks)***

8 (i) Let $f : \mathbb{C} \to \mathbb{R}$ be the map given by $f(z) = \Re(z)$, the real part of z, i.e. $f(a + ib) = a$ (where $a, b \in \mathbb{R}$). Prove that f is a linear map of $\mathbb{R}$-vector spaces. Is it a ring homomorphism? Justify your answers. ***(4 marks)***

(ii) Consider the First Isomorphism Theorem for f as a linear map of $\mathbb{R}$-vector spaces. What does it tell us in this instance? ***(3 marks)***

9 Let V be an $\mathbb{R}$-vector space, with an inner product $\langle , \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from V to V).

(i) What does it mean for T to be *self-adjoint* with respect to $\langle , \rangle$? ***(1 mark)***

(ii) Suppose that T is self-adjoint with respect to $\langle , \rangle$, and that v_1, v_2 are eigenvectors for T, with eigenvalues λ_1, λ_2 respectively. (So $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, with v_1, v_2 non-zero.) Prove that if $\lambda_1 \neq \lambda_2$ then v_1 and v_2 are orthogonal with respect to $\langle , \rangle$. ***(2 marks)***

(iii) Consider $\mathbb{R}^2$ with the usual dot product. We are given that a certain $T \in L(V)$ is self-adjoint, that $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}$ and that 3 is an eigenvalue of T. Find an eigenvector for the eigenvalue 3, a basis with respect to which T is represented by the matrix $\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$, and the matrix C representing T with respect to the standard basis $\{e_1, e_2\}$. ***(4 marks)***

End of Question Paper