The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **2017–2018**

Analysis **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Give precise definitions of what it means for a non–empty set A of real numbers to be (a) *bounded above*, (b) *bounded below*. **(2 marks)**

Prove that the set A is bounded below if and only if $-A$ is bounded above, where

$$-A = \{-a; a \in A\}.$$

(2 marks)

(ii) Consider the set

$$A = (-40, -7) \cup \{-1, \sqrt{\pi}, e\} \cup [3, \sqrt{11}) \cup \{267\}.$$

(a) Does it have a maximum element? If so, what is it?

(b) Does it have a minimum element? If so, what is it?

(c) Does it have a supremum? If so, what is it?

(d) Does it have a infimum? If so, what is it? **(4 marks)**

(iii) Write down the set $-A$, where A is as in (ii), and then answer all the questions (a) to (d) for this set. **(5 marks)**

(iv) (a) Carefully write down the precise mathematical definition of what it means for a sequence of real numbers (a_n) to converge to a limit $l \in \mathbb{R}$. **(1 mark)**

(b) Consider the sequence (a_n) whose nth term is $\frac{1}{7}\left(\frac{3}{14} - \frac{5}{n^2}\right)$. First guess the limit of this sequence, and then give a careful proof that the sequence converges to this limit. Your proof should make use of ϵ, N and the Archimedean property of the real numbers. **(8 marks)**

(c) Which of the following statements about the sequence in (b) are correct? It is bounded above, bounded below, monotonic increasing, monotonic decreasing? **(3 marks)**

2 (i) Using sequences, write down the precise mathematical definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ with domain D_f to have a limit l at a point $a \in \mathbb{R}$. **(2 marks)**

What additional requirements are needed for f to be continuous at a? **(1 mark)**

(ii) (a) Let a and b be non–negative real numbers. Carefully derive the inequality

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

(2 marks)

[Hint: You may use the result that for non–negative real numbers x and y,

$$y \geq x \text{ if and only if } \sqrt{y} \geq \sqrt{x}.]$$

(b) Use the result of (a) to show that

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|},$$

for all $a, b \geq 0$. **(4 marks)**

[Hint: Write $a = |a| = |(a-b)+b|$.]

(c) Show that if (x_n) is a sequence of non–negative real numbers, that converges to a non–negative real number x, then the sequence whose nth term is $\sqrt{x_n}$ converges to $\sqrt{x}$. **(3 marks)**

[Hint: Use ϵ and N, together with the result of (b)]

(d) Given a function $f : \mathbb{R} \to [0, \infty)$ with domain D_f for which $\lim_{x\to a} f(x) = l$, show that $\lim_{x\to a} \sqrt{f(x)} = \sqrt{l}$. **(2 marks)**

(e) Show that if the function f of (d) is continuous at $a \in D_f$, then so is $\sqrt{f}$, where $\sqrt{f}(x) = \sqrt{f(x)}$, for all $x \in D_f$. **(2 marks)**

(f) Using the results just obtained, explain why the function $x \mapsto \sqrt{1 + 4x + 9x^2 + 2x^4}$ is continuous at the point $x = 1$. Hence obtain $\lim_{x\to 1} \sqrt{1 + 4x + 9x^2 + 2x^4}$. **(3 marks)**

(iii) (a) The *intermediate value theorem* applies to a certain type of function $f : [a, b] \to \mathbb{R}$ for which $f(b) < 0$ and $f(a) > 0$. Give a condition on the function for the theorem to hold, and also write down the conclusion of the theorem. **(2 marks)**

(b) Using the result of (a), show that if $g : [a, b] \to \mathbb{R}$ is continuous with $g(b) < g(a)$, then for each $\gamma \in (g(b), g(a))$ there exists $c \in (a, b)$ so that $g(c) = \gamma$. What does this tell us about the relationship between the range of the function g and the interval $[g(b), g(a)]$? **(4 marks)**

3 (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

For each of the following statements, say whether it is true or false, carefully justifying your answer. You may use standard facts about polynomials.

(a) The function f is continuous at every $x \in \mathbb{R}$. **(3 marks)**

(b) The function f is differentiable at every $x \in \mathbb{R}$. **(4 marks)**

(c) The derivative function f' has domain $\mathbb{R}$ and is continuous at every $x \in \mathbb{R}$. **(3 marks)**

(d) The derivative function f' has domain $\mathbb{R}$ and is differentiable at every $x \in \mathbb{R}$. **(3 marks)**

(ii) (a) State the *mean value theorem.* **(2 marks)**

(b) Let $f : \mathbb{R} \to (0, \infty)$ be a differentiable function such that

$$f'(x) = -f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Show, by using the mean value theorem, that f is strictly monotonic decreasing. **(2 marks)**

(c) Let $f : [0, 1] \to \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in [0, 1]$. Show that there exists at most one $c \in [0, 1]$ such that $f(c) = c$. **(3 marks)**

(iii) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series.

(a) Explain what it means to say that this power series has radius of convergence R, with $R > 0$. **(2 marks)**

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^3}{3n+1} x^n.$$

(3 marks)

4 (i) (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define the *lower integral*, $L\int_a^b f(t)\,dt$, and the *upper integral*, $U\int_a^b f(t)\,dt$. Say what it means for the function to be *Riemann integrable* and define the Riemann integral of f in that case. **(4 marks)**

(b) Consider the functions $r : [0,3] \to \mathbb{R}$ and $s : [0,3] \to \mathbb{R}$, given by

$$r = \chi_{[0,1)} + 7\chi_{[1,2)} + 7^4\chi_{[2,3]},$$
$$s = 7\chi_{[0,1)} + 7^4\chi_{[1,2)} + 7^9\chi_{[2,3]}.$$

Calculate

$$\int_0^3 r(t)\,dt, \qquad \text{and} \qquad \int_0^3 s(t)\,dt.$$

(4 marks)

(c) Let $f : [0,3] \to \mathbb{R}$ be a monotonic increasing continuous function, such that $f(0) = 1$, $f(1) = 7$, $f(2) = 7^4$ and $f(3) = 7^9$. Show that

$$1 + 7 + 7^4 \le \int_0^3 f(t)\,dt \le 7 + 7^4 + 7^9.$$

(4 marks)

(ii) For $n \ge 1$, define a function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} 1, & \text{if } 0 \le t < \frac{1}{2} - \frac{1}{2^{n+1}}, \\ 2^n(1-2t), & \text{if } \frac{1}{2} - \frac{1}{2^{n+1}} \le t < \frac{1}{2}, \\ 2^n(2t-1), & \text{if } \frac{1}{2} \le t < \frac{1}{2} + \frac{1}{2^{n+1}}, \\ 1, & \text{if } \frac{1}{2} - \frac{1}{2^{n+1}} \le t \le 1. \end{cases}$$

(a) Sketch the graphs of f_1, f_2 and f_3. **(3 marks)**

(b) Is f_n continuous? Justify your answer. **(4 marks)**

(c) Show that the sequence (f_n) converges pointwise, and say what the limit function f is. **(3 marks)**

(d) Does the sequence (f_n) converge uniformly? Justify your answer. **(3 marks)**

End of Question Paper