SCHOOL OF MATHEMATICS AND STATISTICS

Analysis

2017–2018

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.
(i) Give precise definitions of what it means for a non-empty set $A$ of real numbers to be (a) bounded above, (b) bounded below. (2 marks)

Prove that the set $A$ is bounded below if and only if $-A$ is bounded above, where

$$-A = \{-a; a \in A\}.$$ (2 marks)

(ii) Consider the set

$$A = (-40, -7) \cup \{-1, \sqrt{\pi}, e\} \cup [3, \sqrt{11}) \cup \{267\}.$$ (4 marks)

(a) Does it have a maximum element? If so, what is it?
(b) Does it have a minimum element? If so, what is it?
(c) Does it have a supremum? If so, what is it?
(d) Does it have a infimum? If so, what is it?

(iii) Write down the set $-A$, where $A$ is as in (ii), and then answer all the questions (a) to (d) for this set. (5 marks)

(iv) (a) Carefully write down the precise mathematical definition of what it means for a sequence of real numbers $(a_n)$ to converge to a limit $l \in \mathbb{R}$. (1 mark)

(b) Consider the sequence $(a_n)$ whose $n$th term is $\frac{1}{7} \left( \frac{3}{14} - \frac{5}{n^2} \right)$. First guess the limit of this sequence, and then give a careful proof that the sequence converges to this limit. Your proof should make use of $\epsilon, N$ and the Archimedean property of the real numbers. (8 marks)

(c) Which of the following statements about the sequence in (b) are correct? It is bounded above, bounded below, monotonic increasing, monotonic decreasing? (3 marks)
2  (i) Using sequences, write down the precise mathematical definition of what it means for a function \( f : \mathbb{R} \to \mathbb{R} \) with domain \( D_f \) to have a limit \( l \) at a point \( a \in \mathbb{R} \). (2 marks)

What additional requirements are needed for \( f \) to be continuous at \( a \)? (1 mark)

(ii)  (a) Let \( a \) and \( b \) be non-negative real numbers. Carefully derive the inequality

\[
\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}.
\]

(2 marks)

[Hint: You may use the result that for non-negative real numbers \( x \) and \( y \),
\[ y \geq x \text{ if and only if } \sqrt{y} \geq \sqrt{x}. \]

(b) Use the result of (a) to show that

\[
|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|},
\]

for all \( a, b \geq 0 \). (4 marks)

[Hint: Write \( a = |a| = |(a - b) + b| \).]

(c) Show that if \( (x_n) \) is a sequence of non-negative real numbers, that converges to a non-negative real number \( x \), then the sequence whose \( n \)th term is \( \sqrt{x_n} \) converges to \( \sqrt{x} \). (3 marks)

[Hint: Use \( \epsilon \) and \( N \), together with the result of (b)].

(d) Given a function \( f : \mathbb{R} \to [0, \infty) \) with domain \( D_f \) for which \( \lim_{x \to a} f(x) = l \), show that \( \lim_{x \to a} \sqrt{f(x)} = \sqrt{l} \). (2 marks)

(e) Show that if the function \( f \) of (d) is continuous at \( a \in D_f \), then so is \( \sqrt{f} \), where \( \sqrt{f}(x) = \sqrt{f(x)} \), for all \( x \in D_f \). (2 marks)

(f) Using the results just obtained, explain why the function \( x \mapsto \sqrt{1 + 4x + 9x^2 + 2x^4} \) is continuous at the point \( x = 1 \). Hence obtain \( \lim_{x \to 1} \sqrt{1 + 4x + 9x^2 + 2x^4} \). (3 marks)

(iii)  (a) The intermediate value theorem applies to a certain type of function \( f : [a, b] \to \mathbb{R} \) for which \( f(b) < 0 \) and \( f(a) > 0 \). Give a condition on the function for the theorem to hold, and also write down the conclusion of the theorem. (2 marks)

(b) Using the result of (a), show that if \( g : [a, b] \to \mathbb{R} \) is continuous with \( g(b) < g(a) \), then for each \( \gamma \in (g(b), g(a)) \) there exists \( c \in (a, b) \) so that \( g(c) = \gamma \). What does this tell us about the relationship between the range of the function \( g \) and the interval \( [g(b), g(a)] \)? (4 marks)
Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by
\[
f(x) = \begin{cases} 
x^2 & \text{if } x < 0, \\
0 & \text{if } x \geq 0.
\end{cases}
\]

For each of the following statements, say whether it is true or false, carefully justifying your answer. You may use standard facts about polynomials.

(a) The function \( f \) is continuous at every \( x \in \mathbb{R} \). (3 marks)

(b) The function \( f \) is differentiable at every \( x \in \mathbb{R} \). (4 marks)

(c) The derivative function \( f' \) has domain \( \mathbb{R} \) and is continuous at every \( x \in \mathbb{R} \). (3 marks)

(d) The derivative function \( f' \) has domain \( \mathbb{R} \) and is differentiable at every \( x \in \mathbb{R} \). (3 marks)

(ii) (a) State the mean value theorem. (2 marks)

(b) Let \( f : \mathbb{R} \to (0, \infty) \) be a differentiable function such that
\[
f'(x) = -f(x) \quad \text{for all } x \in \mathbb{R}.
\]
Show, by using the mean value theorem, that \( f \) is strictly monotonic decreasing. (2 marks)

(c) Let \( f : [0, 1] \to \mathbb{R} \) be differentiable such that \( |f'(x)| < 1 \) for all \( x \in [0, 1] \).
Show that there exists at most one \( c \in [0, 1] \) such that \( f(c) = c \). (3 marks)

(iii) Let \( \sum_{n=0}^{\infty} a_n x^n \) be a power series.

(a) Explain what it means to say that this power series has radius of convergence \( R \), with \( R > 0 \). (2 marks)

(b) Find the radius of convergence of the power series
\[
\sum_{n=0}^{\infty} \frac{n^3}{3n + 1} x^n.
\] (3 marks)
(a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the lower integral, $L \int_a^b f(t) \, dt$, and the upper integral, $U \int_a^b f(t) \, dt$. Say what it means for the function to be Riemann integrable and define the Riemann integral of $f$ in that case. 

(b) Consider the functions $r : [0, 3] \to \mathbb{R}$ and $s : [0, 3] \to \mathbb{R}$, given by

$$r = \chi_{[0,1]} + 7\chi_{[1,2]} + 7^4\chi_{[2,3]},$$
$$s = 7\chi_{[0,1]} + 7^4\chi_{[1,2]} + 7^9\chi_{[2,3]}.$$ 

Calculate

$$\int_0^3 r(t) \, dt,$$
$$\int_0^3 s(t) \, dt.$$ 

(c) Let $f : [0, 3] \to \mathbb{R}$ be a monotonic increasing continuous function, such that $f(0) = 1$, $f(1) = 7$, $f(2) = 7^4$ and $f(3) = 7^9$. Show that

$$1 + 7 + 7^4 \leq \int_0^3 f(t) \, dt \leq 7 + 7^4 + 7^9.$$ 

(ii) For $n \geq 1$, define a function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} 
1, & \text{if } 0 \leq t < \frac{1}{2} - \frac{1}{2^{n+1}}, \\
2^n(1 - 2t), & \text{if } \frac{1}{2} - \frac{1}{2^{n+1}} \leq t < \frac{1}{2}, \\
2^n(2t - 1), & \text{if } \frac{1}{2} \leq t < \frac{1}{2} + \frac{1}{2^{n+1}}, \\
1, & \text{if } \frac{1}{2} - \frac{1}{2^{n+1}} \leq t \leq 1.
\end{cases}$$

(a) Sketch the graphs of $f_1, f_2$ and $f_3$. 

(b) Is $f_n$ continuous? Justify your answer. 

(c) Show that the sequence $(f_n)$ converges pointwise, and say what the limit function $f$ is. 

(d) Does the sequence $(f_n)$ converge uniformly? Justify your answer.