The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2017–2018**

Differential Equations **2.5 hours**

*Attempt **ALL** questions. The allocation of marks is shown in brackets. Total marks 100.*

1 (i) Sketch the phase line of $\dot{u} = \cos(2\pi u)$, for $-1 \leq u \leq 1$. State **all** the equilibrium points for $u \in \mathbb{R}$ (note: **not just** those for which $-1 \leq u \leq 1$). Which equilibrium points are stable and which are unstable? ***(3 marks)***

(ii) Consider the following system of linear ordinary differential equations (ODEs)

$$\dot{u} = -2u + av,$$
$$\dot{v} = au - 2v,$$

where a is a real number. Determine the nature (e.g. node, saddle, spiral, centre etc.) and stability of the equilibrium point at (0,0) in the cases $a = 1$ and $a = 3$. ***(3 marks)***

For which values of a is the equilibrium point at the origin stable? ***(2 marks)***

(iii) The ODE $\ddot{x} + \dot{x} + \sin(x) = 0$ models a damped oscillator. Show that this can be written as the following planar system of ODEs ***(1 mark)***

$$\dot{x} = y, \quad (1)$$
$$\dot{y} = -y - \sin(x). \quad (2)$$

Find the equilibrium points of the system in equations (1-2). ***(2 marks)***

Write down the Jacobian of the system in equations (1-2). ***(2 marks)***

Determine the nature (e.g. spiral, node, centre etc.) and stability (i.e. stable or unstable) of the equilibrium points. ***(3 marks)***

Sketch the nullclines of equations (1-2) for $-\pi \leq x \leq 3\pi$ and $-3/2 \leq y \leq 3/2$. ***(3 marks)***

On a **separate diagram**, sketch the phase portrait of the system for $-\pi \leq x \leq 3\pi$ and $-3/2 \leq y \leq 3/2$. Include sufficiently many trajectories such that the long-term behaviour of the system from any starting-point is qualitatively clear. ***(6 marks)***

2 (i) Find the solution of the equation $y'' + y = 0$ subject to the boundary conditions $y(0) = 0$, $y(\pi/2) = 1$. ***(4 marks)***

(ii) If we look for solutions of the equation $y'' - 2xy' + 2y = 0$ of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with $y(0) = y'(0) = 1$, what are the values of a_n for $n = 0, 1, 2, 3$?

(6 marks)

(iii) Show that the normal form of

$$x^2 y'' + 3xy' + y = 0, \tag{3}$$

is

$$4x^2 u'' + u = 0, \tag{4}$$

where $y(x) = u(x)v(x)$ and $v(x)$ is the solution of a first order ODE.

(7 marks)

(iv) Show that $u(x) = Ax^{1/2}$ is a solution of equation (4). ***(2 marks)***

Use Reduction of Order to find another (linearly independent) solution of equation (4). ***(5 marks)***

(v) Use the results from parts (iii) and (iv) to show that $y(x) = x^{-1}[a + b\ln(x)]$ is a solution of equation (3), where a and b are arbitrary constants.

(1 mark)

3 Let $T(x,t) = F(x)G(t)$ be a separable solution of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{5}$$

for $0 < x < L$ and $t \geq 0$, where $\kappa > 0$ and $L > 0$ are constants.

Show that the functions $F(x)$ and $G(t)$ satisfy the following ordinary differential equations:

$$\frac{d^2 F}{dx^2} - \alpha F = 0, \qquad \frac{dG}{dt} = \kappa \alpha G,$$

where α is an arbitrary constant. ***(3 marks)***

Find the general solution for $G(t)$. ***(2 marks)***

If, in addition, the function $T(x,t)$ is subject to the boundary conditions

$$\frac{\partial T}{\partial x}(0,t) = 0, \qquad T(L,t) = 0, \tag{6}$$

write down the boundary conditions that must be satisfied by the function $F(x)$. ***(1 mark)***

If $\alpha \geq 0$, show that the only separable solution of the heat equation (5) subject to the boundary conditions (6) is the trivial solution $T(x,t) \equiv 0$. ***(7 marks)***

If $\alpha < 0$, find all nontrivial separable solutions of the heat equation (5) subject to the boundary conditions (6). ***(6 marks)***

Show that the principle of superposition applies to solutions of the heat equation (5) subject to the boundary conditions (6) and explain why the general solution of the heat equation (5) subject to the boundary conditions (6) is

$$T(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{[2n+1]\,\pi x}{2L}\right) \exp\left(-\frac{[2n+1]^2\,\pi^2 \kappa t}{4L^2}\right).$$

(3 marks)

Find the solution of the heat equation (5) subject to the boundary conditions (6) and the initial condition

$$T(x,0) = \cos\left(\frac{\pi x}{2L}\right) + \cos\left(\frac{5\pi x}{2L}\right).$$

(3 marks)

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QUESTION FOUR ON NEXT PAGE

4 (i) Find the characteristics of the first order PDE

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0, \tag{7}$$

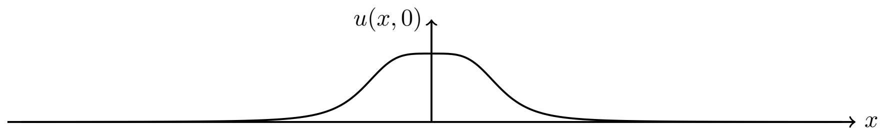
where V is a constant, independent of x and t. ***(2 marks)***

Find the general solution $u(x,t)$ of the PDE (7). ***(5 marks)***

At time $t = 0$, the function $u(x,t)$, as well as being a solution of the PDE (7), also satisfies the initial condition

$$u(x,0) = \frac{1}{1+x^4}. \tag{8}$$

The function $u(x,0)$ is shown below:



Sketch the function $u(x,t)$ for a fixed $t > 0$ if $V > 0$. ***(2 marks)***

(ii) Show that the change of variables $(x,t) \to (\xi(x,t), \eta(x,t))$ with

$$\xi(x,t) = x + Vt, \qquad \eta(x,t) = x - Vt, \tag{9}$$

where $V > 0$ is a positive constant, is invertible for all x and t. ***(2 marks)***

Find the resulting PDE if the change of variables (9) is performed on the PDE

$$\frac{\partial q}{\partial t} - V\frac{\partial q}{\partial x} = \frac{4V\,(x-Vt)^3}{\left[1+(x-Vt)^4\right]^2}. \tag{10}$$

(3 marks)

Hence find the general solution of the PDE (10). ***(2 marks)***

4 (continued)

(iii) Show that the function $u(x,t)$, given in terms of another function $q(x,t)$ by

$$u(x,t) = \frac{\partial q}{\partial t} - V\frac{\partial q}{\partial x},$$

satisfies the PDE (7) if the function $q(x,t)$ satisfies the second order linear PDE

$$V^2\frac{\partial^2 q}{\partial x^2} - \frac{\partial^2 q}{\partial t^2} = 0, \tag{11}$$

where $V > 0$ is a constant. ***(1 mark)***

At time $t = 0$, the function $q(x,t)$ satisfies the initial conditions

$$q(x,0) = \frac{1}{1+x^4}, \qquad \frac{\partial q}{\partial t}(x,0) = 0. \tag{12}$$

Using your answer to part (i), find the solution of (7) for $u(x,t)$.
(3 marks)

Using your answer to part (ii), hence find the solution $q(x,t)$ of the PDE (11) subject to the initial conditions (12). ***(4 marks)***

Sketch the solution $q(x,t)$ of the PDE (11) subject to the initial conditions (12), at a fixed time $t > 0$. ***(1 mark)***

End of Question Paper